4

Synthesis of Networks

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4.1 Introduction

Synthesis of electrical networks is an area of electrical engineering in which one attempts to find the network from given specifications. In most cases, this is applied to filters constructed with various elements.

This chapter begins with a few words about history. Before the second world war, the main communication products were radios working with amplitude modulation. Stations transmitted on various frequencies, and it was necessary to get only the desired one and suppress all the others. This led to the development of various filters. Two men contributed fundamentally to the filter design theory: Darlington, from the United States, and Cauer, from Germany.

As time went by, it was discovered that inductor capacitor (LC) filters were not suitable for many applications, especially in low frequency regions where inductors are large and heavy. An idea occurred: replace the inductors by active networks composed of amplifiers with feedback by means of resistors and/or capacitors. This was the era of active networks, approximately in the 1960s to 1980s of the 20th century.

Technological advances and miniaturization led to integrated circuits, with the attempt to get complete filters on a chip. Here it turned out that using resistors was not convenient, and a new idea emerged of switched capacitor networks. In these networks, the resistors are replaced by rapidly switched capacitors that act approximately as resistors. In such networks, we have only capacitors and transistors work as amplifiers or switches. Capacitors and transistors are suitable for integration.

New developments continue to find their ways. This Chapter limits explanations mostly to LC filters. Switched capacitor networks and later theoretical developments are too complex to cover in this short contribution.

Returning now to filter synthesis, this discussion must first clarify the appropriate theoretical tools. Filters are linear devices that allow the use of linear network theories. Specifications of filters are almost always given as frequency domain responses, which leads to the use of Laplace transform. This chapter assumes that the reader is at least somewhat familiar with the concept of the transformation because the discussion deals with the complex plane, the complex variable s, frequency domain responses, poles, and zeros. A sufficient amount of information is given so that a reader can refresh his or her knowledge.

4.2 Elementary Networks

Filters can be built with passive elements that do not need any power supply to retain their properties and with active elements that work only when electrical power is supplied from a battery or from a power supply.

The passive elements are inductors (L), capacitors (C), and resistors (R). Resistance of the resistor is measured in ohms. Its inverse value is a conductance, $G = 1/R$. For inductors and capacitors, we use the Laplace transform; in such a case, we speak about impedances of these elements: $Z_L = sL$ and $Z_C = 1/sC$. The inverse of the impedance is the admittance, $Y_C = sC$ and $Y_L = 1/sL$. The subscripts used here are for clarification only and are usually not used. Symbols for these elements are in Figure 4.1. Such networks have two terminals, sometimes called a port. Notice the voltage signs and positive directions of the currents. This is an important convention because it is also valid for independent voltage and current sources, shown in Figure 4.2. We use the letter E for the independent voltage source and the letter J for the independent current source to distinguish them from voltages and currents anywhere inside the network.

From the above description of a one-port network, we can refer to the concept of a two-port network, usually drawn so that the input port is on the left and the output port is on the right. We can introduce a general symbol for the two-port as

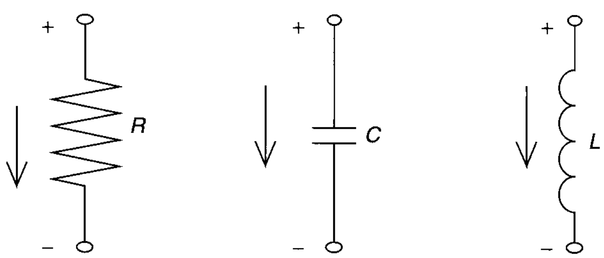


FIGURE 4.1 Symbols for a Resistor, Capacitor, and Inductor

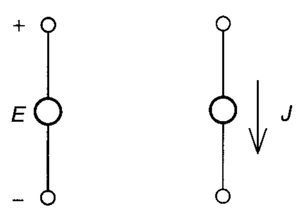


FIGURE 4.2 Symbols for Independent Voltage (E) and Independent Current (J) Source

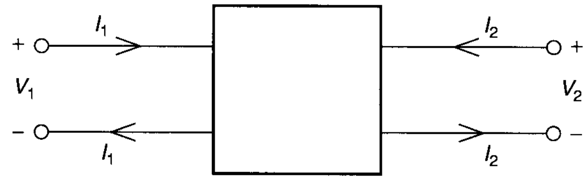


FIGURE 4.3 Symbol for a Two-Port

shown in Figure 4.3. Notice that in such a case we speak about the input voltage across the left one-port, V_1, and the output voltage across the right one-port, V_2. The plus sign is on top and minus sign is at the bottom. We also indicate the currents I_1 and I_2 as shown. Notice how the currents flow into the two-port and how they leave it. It is very important to know these directions.

There exist four elementary two-ports, shown in Figure 4.4. They are the most simplified forms of amplifiers. The voltage-controlled voltage source, *VV*, has its output controlled by the input voltage:

$$V_2 = \mu V_1. \tag{4.1}$$

The μ is a dimensionless constant. Another elementary two-port is a voltage-controlled current source, *VC*, described by the equation:

$$I_2 = g V_1, \tag{4.2}$$

where g is transconductance and has the dimension of a conductance. The third elementary two-port is the current-controlled voltage source, *CV*, defined by:

$$V_2 = r I_1, \tag{4.3}$$

where r represents transresistance and has the dimension of a resistor. The last such two-port is a current-controlled current source, *CC*, defined by:

$$I_2 = \alpha I_1, \tag{4.4}$$

where α is a dimensionless constant.

Any number of elements can be variously connected; if we consider on such a network an input port and an output port, we will have a general two-port. This two-port has four variables, V_1, V_2, I_1, and I_2, as in Figure 4.3. Any two can be selected as independent variables, and the other two will be dependent variables. For instance, we can consider both currents as independent variables, which will make the voltages dependent variables. To couple them in a most general form, we can write the following:

$$\begin{aligned} V_1 &= z_{11} I_1 + z_{12} I_2. \\ V_2 &= z_{21} I_1 + z_{22} I_2. \end{aligned} \tag{4.5}$$

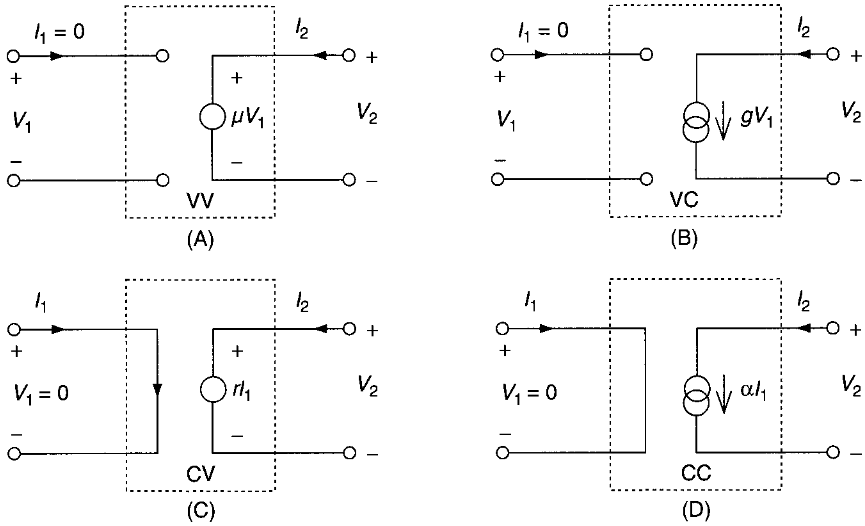


FIGURE 4.4 Simplest Two-Ports

In equation 4.5, the z_{ij} have dimensions of impedances, and we speak about the impedance description of the network. The equations are usually cast into a matrix equation:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{11} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{4.6}$$

Another way of expressing the dependences is to select the voltages as independent variables and the currents as dependent variables and present them in the form of two equations:

$$\begin{aligned} I_1 &= y_{11} V_1 + y_{12} V_2. \\ I_2 &= y_{21} V_1 + y_{22} V_2. \end{aligned} \tag{4.7}$$

This is an admittance description of a two-port, and in matrix form it is:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{4.8}$$

There exist additional possibilities on how to couple the variables. They can be found in any textbook on network theory.

For demonstration, we use the network in Figure 4.5 and derive its z_{ij} parameters. We attach a voltage source V_1 on the left and nothing on the right. This means that $I_2 = 0$, and the voltage V_2 appears in the middle of the network as indicated. We see that $V_1 = (Z_1 + Z_3)I_1$ and $z_{11} = V_1/I_1 = Z_1 + Z_3$. In addition, since the voltage in the middle of the network is V_2, we can write $V_2 = I_1 Z_3$ and $z_{21} = V_2/I_1 = Z_3$. In the next step, we place the voltage source on the right and proceed similarly. The result is:

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix} \tag{4.9}$$

In the next example, we consider the connection of a two-port to a loading resistor, as sketched in Figure 4.6. We wish to find the input impedance of the combination. After we connect them, as indicated by the dashed line, there is the same voltage, V_2, across the second port and across the resistor. The current flowing into the resistor will be $-I_2$ and thus $V_2 = -I_2 R$. Inserting into the second equation of the set (4.5), we obtain this equation:

$$I_2 = \frac{-z_{21}}{R + z_{22}} I_1.$$

Replacing I_2 in the first equation by this result, we get:

$$Z_{in} = V_1/I_1 = \frac{z_{11}R + z_{11}z_{22} - z_{12}z_{21}}{R + z_{22}}. \tag{4.10}$$

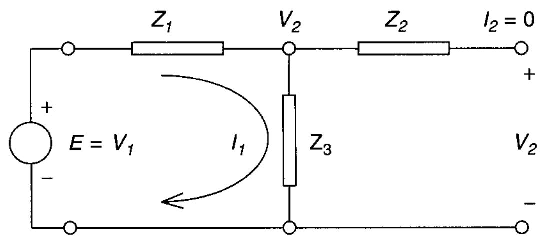


FIGURE 4.5 Finding Z Parameters for the Network

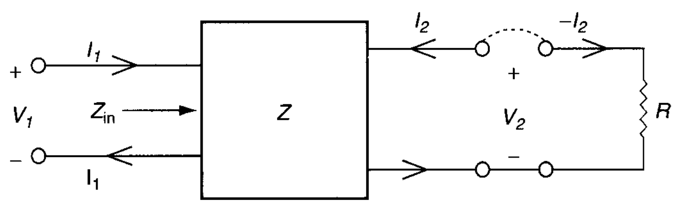


FIGURE 4.6 Obtaining Input Impedance of a Two Port Loaded by a Resistor

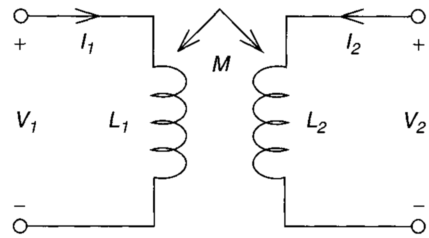


FIGURE 4.7 Symbol for a Technical Transducer

We will conclude this section by introducing two more two-ports, namely the ideal and the technical transformer[5]. The ideal transformer, often used in the synthesis theory, is described by the equations:

$$\begin{aligned} V_2 &= nV_1. \\ I_2 &= -\frac{1}{n}I_1. \end{aligned} \tag{4.11}$$

We note that we cannot put equation 4.11 into any of the two matrix forms that we introduced (impedance or admittance form). A technical transformer is realized by magnetically coupled coils. The device is described by the equations:

$$\begin{aligned} V_1 &= sL_1I_1 + sMI_2. \\ V_2 &= sMI_1 + sL_2I_2 \end{aligned} . \tag{4.12}$$

The L_1 and L_2 are the primary and secondary inductances, and M is the mutual inductance. These variables can be expressed in the impedance form as:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} sL_1 & sM \\ sM & sL_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{4.13}$$

Figure 4.7 Shows the technical transformer.

4.3 Network Functions

In the second section, we introduced two-port networks with their input and output variables. We now introduce the concept of networks with only one source and one output. As an example, take the network in Figure 4.8. The output voltage is:

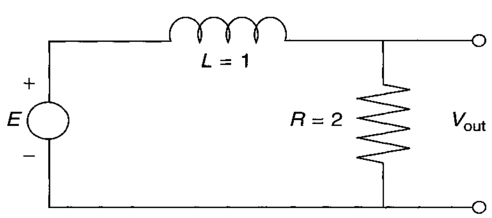


FIGURE 4.8 Introducing a Network Function

$$V_{out} = \frac{2}{s+2}E, \tag{4.14}$$

where E is any independent signal voltage. If we do not consider a specific signal but rather divide the equation by E, we get the voltage transfer function:

$$T_V = \frac{V_{out}}{E} = \frac{2}{s+2}. \tag{4.15}$$

This is one of the possible network functions. Had we chosen a larger network, the ratio would be in the form:

$$T_V = \frac{V_{out}}{E} = \frac{N(s)}{D(s)}, \tag{4.16}$$

where $N(s)$ and $D(s)$ are polynomials in the variable s.

One network may have many network functions. The number depends on where we apply the independent source, which source we select, and where we take the output. To be able to define the network function, we must satisfy two conditions: (1) the network is linear, and (2) there are zero initial conditions on capacitors and inductors.

If these conditions are satisfied, then we define a general network function as the ratio:

$$\text{Network function} = \frac{\text{output}}{\text{input}}. \tag{4.17}$$

Depending on the type of independent input source (voltage or current) and the type of the output (voltage or current), we can define the following network functions:

1. Voltage transfer is $T_V = \frac{V_{out}}{E}$.
2. Current transfer is $T_I = \frac{I_{out}}{J}$.
3. Transfer impedance is $Z_{TR} = \frac{V_{out}}{J}$.
4. Transfer admittance is $Y_{TR} = \frac{I_{out}}{E}$.
5. Input impedance is $Z_{in} = \frac{V_{in}}{J}$.
6. Input admittance is $Y_{in} = \frac{I_{in}}{E}$.

The input impedance is the inverse of the input admittance, but this is not true for the transfer impedances and admittances.

As an example, we take the network in Figure 4.9. Using nodal analysis, we can write the equations:

$$V_1(G_1 + G_2 + sC) - G_2V_2 = J.$$

$$-G_2V_1 + (G_2 + 1/sL)V_2 = 0.$$

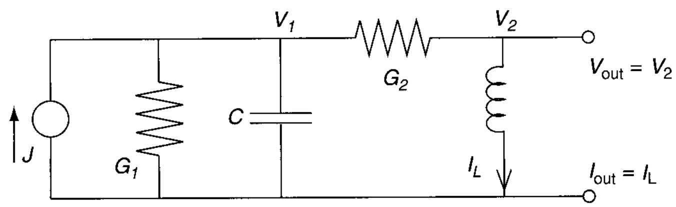


FIGURE 4.9 Finding Two Network Functions of One Network

They can also be written in matrix form:

$$\begin{bmatrix} (G_1 + G_2 + sC) & -G_2 \\ -G_2 & (G_2 + 1/sL) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} J \\ 0 \end{bmatrix}$$

The denominator of the transfer function is the determinant of the system matrix:

$$D = sCG_2 + (G_1 G_2 + C/L) + (G_1 + G_2)/sL.$$

Using Cramer's rule, we obtain:

$$V_{out} = \frac{G_2 J}{D}.$$

Since $I_{out} = V_{out}/sL$, we also get

$$I_{out} = (G_2/sL)/D.$$

We remove the complicated fractions by multiplying the numerator and the denominator by sL, with the result:

$$Z_{TR} = \frac{sLG_2}{s^2 LCG_2 + s(LG_1 G_2 + C) + G_1 + G_2}.$$

$$T_I = \frac{G_2}{s^2 LCG_2 + s(LG_1 G_2 + C) + G_1 + G_2}.$$

This example demonstrated that the network functions are ratios of polynomials in the variable s. If the elements are given by their numerical values, we can find roots of such polynomials. Roots of the numerator polynomial are called **zeros**, and roots of the denominator are called **poles**. From mathematics, we know that a polynomial has many zeros just as the highest power of the s variable. In the above example, the denominator is a polynomial of second degree, and the network has two poles. Since the numerator in this example is only a constant, the network will not have any (finite) zeros.

In general, the poles (or zeros) can be real or can appear in complex conjugate pairs. We can draw them in the complex plane, and we can use crosses to mark the poles and use small circles to mark the zeros. Figure 4.10 shows the pole-zero plot of a network with one real and two complex conjugate poles and with two purely imaginary zeros.

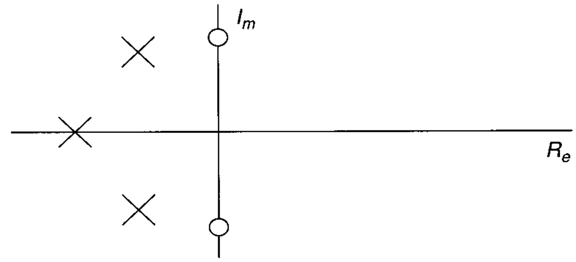


FIGURE 4.10 Pole and Zero Positions of Some Low-Pass Filter

4.4 Frequency Domain Responses

Let us now consider any one of the above network functions and denote it by the letter F. The function will be the ratio of two polynomials in the variables s, $N(s)$, and $D(s)$. If we use a root-finding method, we can also get the poles, p_i, and zeros, z_i. As a result, for a general network function, we can use one of the two following forms:

$$F(s) = \frac{N(s)}{D(s)} = K \frac{\prod (s - z_i)}{\prod (s - p_j)}. \tag{4.18}$$

To obtain various network responses, we substitute

$$s = j\omega. \tag{4.19}$$

$$\omega = 2\pi f = \frac{2\pi}{T}. \tag{4.20}$$

In equations 4.19 and 4.20, f is the frequency of the signal we applied, and $f = 1/T$. Using one frequency and substituting it into equation 4.18, we obtain one complex number with a real and imaginary part:

$$F = x + jy = \mathrm{Re}^{j\phi} \tag{4.21}$$

Here

$$R = \sqrt{x^2 + y^2} \tag{4.22}$$

and

$$\tan\phi = \frac{y}{x}. \tag{4.23}$$

This is sketched in Figure 4.11.

If we evaluate the absolute value or the angles for many frequencies and plot the frequency on the horizontal axis and the resulting values on the vertical axis, we obtain the amplitude and the phase responses of the network.

Calculation of the frequency responses is always done by a computer. We assume that the reader knows how this calculation is done, but we show the responses for the filter in Figure 4.12.

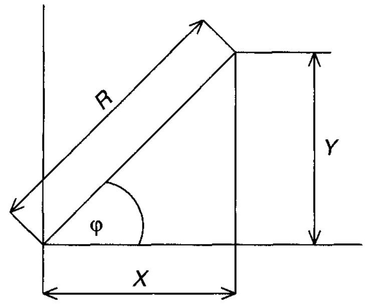


FIGURE 4.11 Finding the Absolute Value and Phase in Complex Plane

We have several possibilities for plotting them: the horizontal axis can be linear or logarithmic, the amplitude responses can be in absolute values, $|F_i| = |F(j\omega_i)|$, or $|F_{i_1 \mathrm{dB}}| = 20 \log |F_i|$ (in decibels, which is a logarithmic scale). The four possible plots of the same responses are in Figures 4.13(A) through 4.13(D). Clearly, selecting a correct scale and plot is very important for proper presentation and understanding.

The phase responses can also be plotted, but we have a couple of problems: the tangent function is periodic, and the plots experience jumps. For this reason, the phase response is rarely plotted, and we prefer to use the group delay, defined by:

$$\tau = -\frac{d\phi}{d\omega}. \tag{4.24}$$

Plot of the group delay for the filter in Figure 4.12 is in Figure 4.14. We see that the amplitude response approximates fairly well a constant in the passband, but the group delay has a large peak. Not every program has a built-in evaluation of the group delay, but there is a remedy: every computer has a polynomial root-finding routine. We can calculate the poles and zeros and use the second formula in equation 4.18. Let there be M zeros

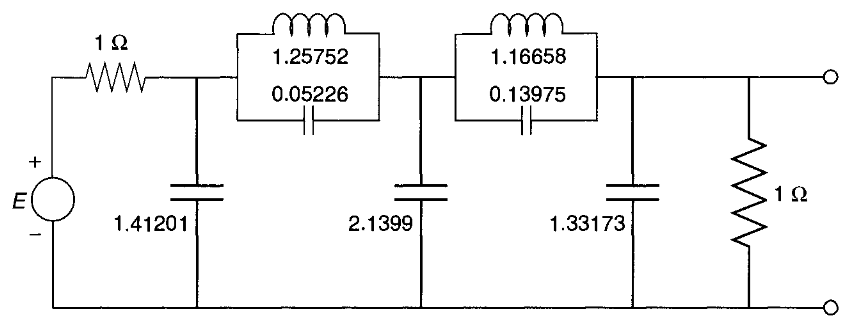


FIGURE 4.12 Example of a Low-Pass Filter with Transfer Zeros

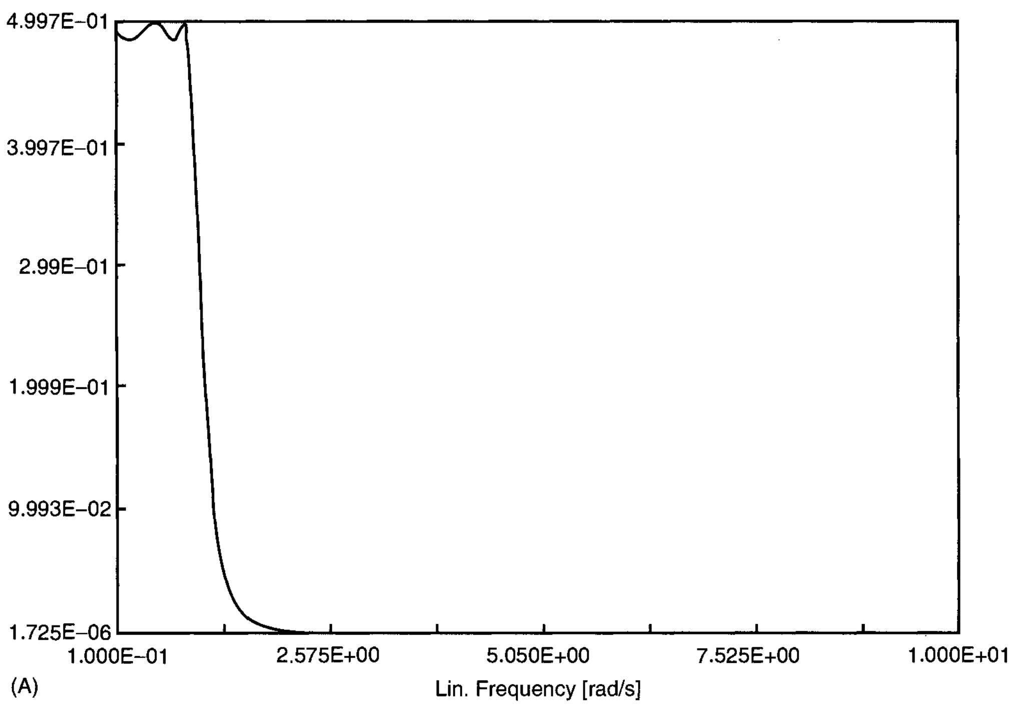
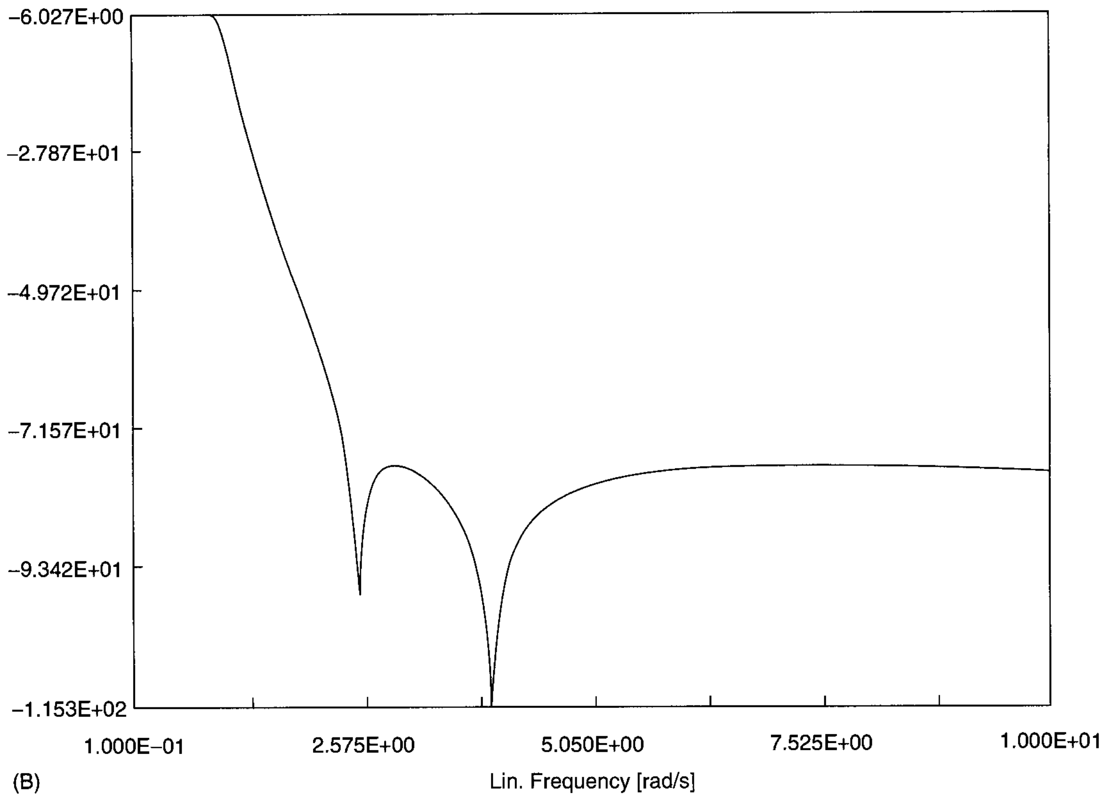


FIGURE 4.13 (A) LC Filter with Transmission Zeros AC Analysis, Magnitude of V_4, Linear Frequency Scale



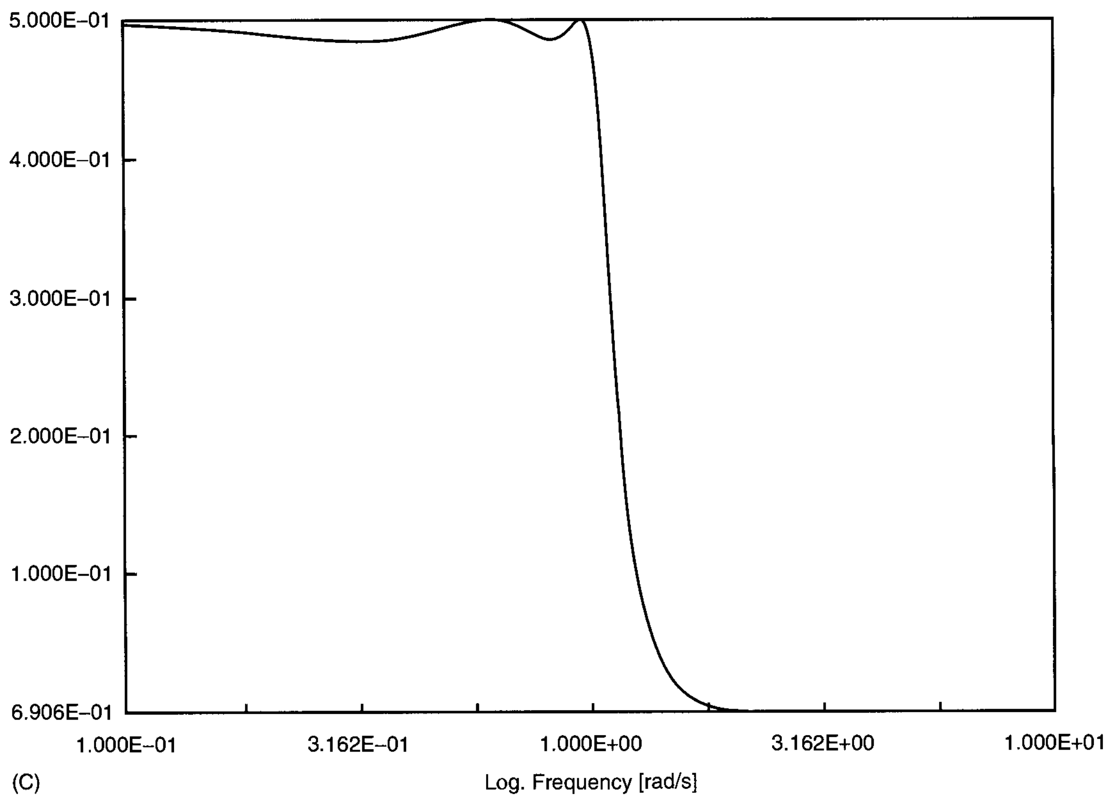


FIGURE 4.13 (cont'd) (B) Same Filter, dB of V_4, Linear Frequency Scale (C) Same Filter, Magnitude of V_4, Logarithmic Frequency Scale

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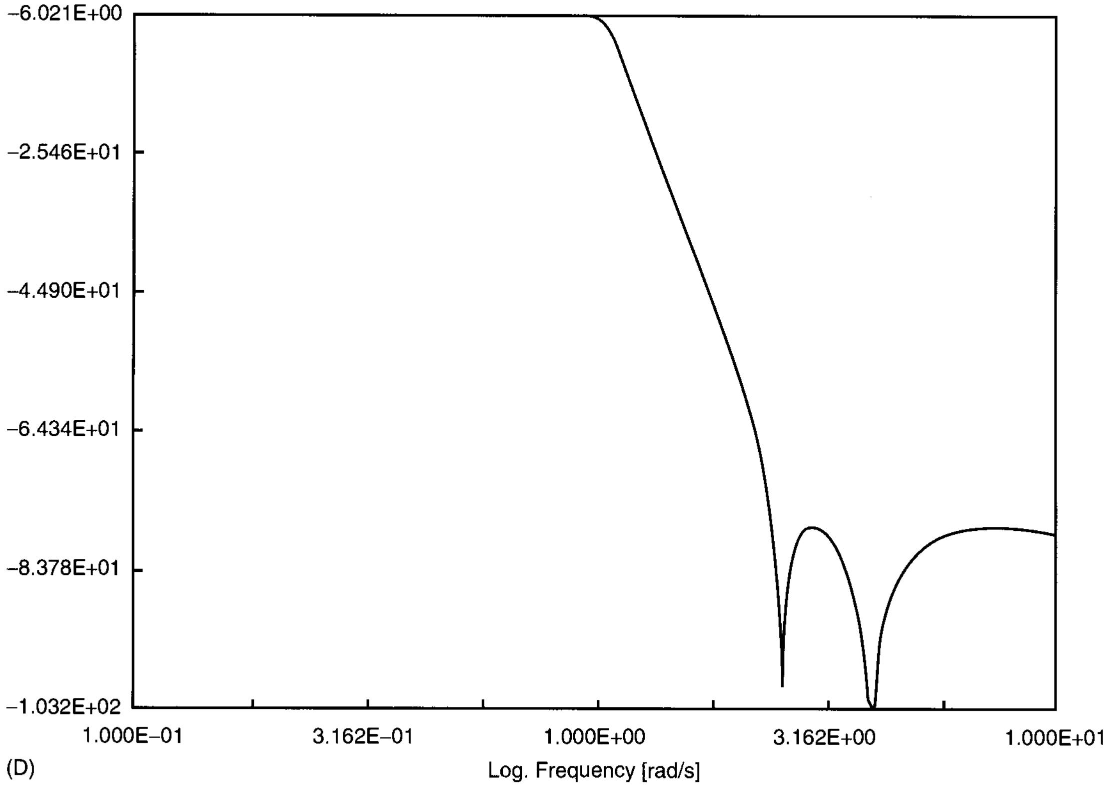


FIGURE 4.13 (cont'd) (D) Same Filter, dB of V_4, k Logarithmic Scale

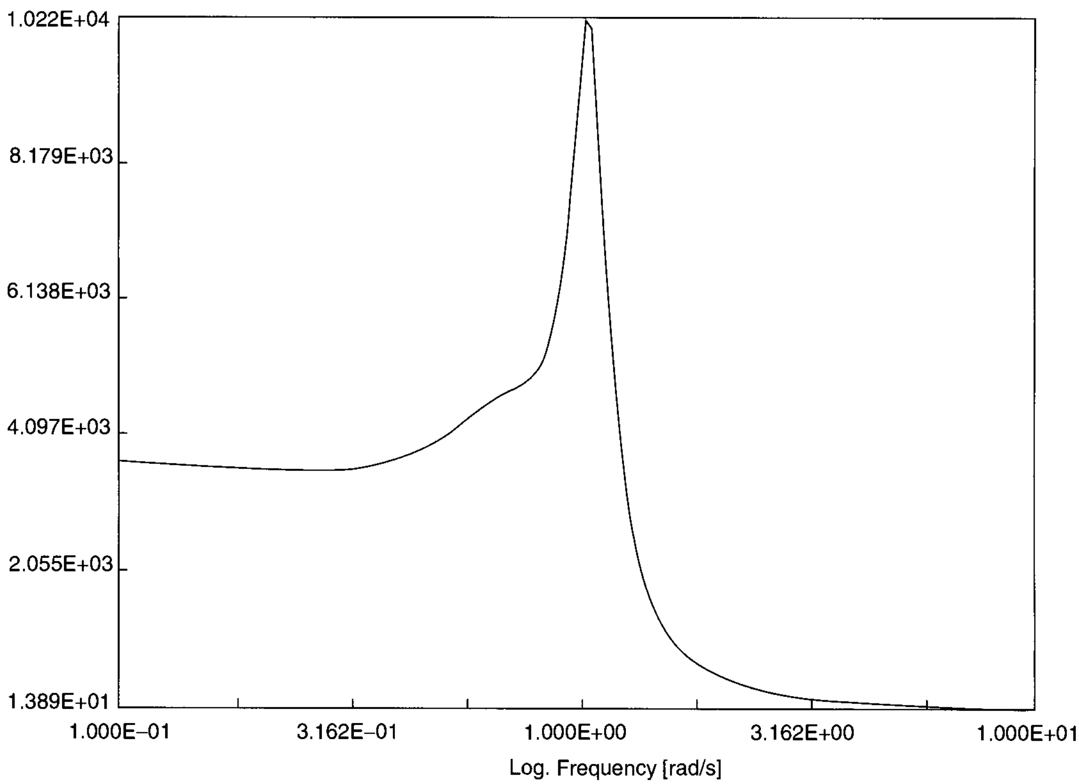


FIGURE 4.14 LC Filter, Group Delay of V_4, in ms, Logarithmic Frequency Scale

$z_i = \alpha_i + \beta_i$ and N poles $p_i = \gamma_i + \delta_i$. Substituting $s = j\omega$ and taking the absolute value, we arrive at the formula:

$$|F| = K\left[\frac{\prod_{i=1}^{M}[\alpha_i^2 + (\omega - \beta_i)^2]}{\prod_{i=1}^{N}[\gamma_i^2 + (\omega - \delta_i)^2]}\right]^{1/2} \quad (4.25)$$

The phase response will be as follows:

$$\Phi = \sum_{i=1}^{M} \arctan\frac{\beta_i - \omega}{\alpha_i} - \sum_{i=1}^{N} \arctan\frac{\delta_i - \omega}{\gamma_i}. \quad (4.26)$$

Differentiating equation 4.26 with respect to ω_o provides the formula for the group delay:

$$\tau(\omega) = \sum_{i=1}^{M}\frac{\alpha_i}{\alpha_i^2 + (\omega - \beta_i)^2} - \sum_{i=1}^{N}\frac{\gamma_i}{\gamma_i^2 + (\omega - \delta)^2}. \quad (4.27)$$

All formulas are easily programmed and plotted.

4.5 Normalization and Scaling

Normalization is one of the most useful concepts in linear networks. It allows us to reduce one frequency and the network impedance to unit values without losing any information. Consider first the scaling of impedances. If we increase (decrease) the impedance of every element of a filter described by its voltage or current transfer function, the input–output relationship should not change. It is thus customary to apply scaling, which reduces one (any) resistor to the value of 1. We can also scale the frequency so that the cutoff frequency of a low-pass filter or the center frequency of a band-pass filter is reduced to 1 rad/s.

To derive the necessary formulas, we will denote the scaled values by the subscript s and leave the values to be used for realization without subscript. Impedance scaling means that the impedance of every network element must be scaled by the same constant, k. This immediately leads to $R_s = R/k$. For frequency normalization, we introduce the formula:

$$\omega_s = \frac{\omega}{\omega_0}, \quad (4.28)$$

where ω_0 is a constant by which we wish to scale the frequency. Now consider the impedance of a scaled inductor by writing:

$$Z_{L,s} = j\omega\frac{L}{k} = j\frac{\omega}{\omega_0}\frac{\omega_0 L}{k} = j\omega_s(\frac{\omega_0 L}{k}).$$

Similarly, for a capacitor we obtain:

$$Z_{C,s} = \frac{1}{j\omega Ck} = \frac{1}{j\frac{\omega}{\omega_0}(\omega_0 Ck)} = \frac{1}{j\omega_S C_S}.$$

The results can be collected in the following formulas:

$$R_S = \frac{R}{k}, \quad G_S = Gk.$$
$$L_S = \frac{L\omega_0}{k}. \quad (4.29)$$
$$C_S = C\omega_0 k.$$

If the network has amplifiers, then:

- VV and CC remain unchanged;
- VC transconductance g is multiplied by k; and
- CV transresistance r is divided by k.

The scaling makes it possible to provide numerous tables of filters, all normalized so that one resistor is equal to 1 Ω and some frequency is normalized to $\omega_S = 1$. From such tables, the reader selects a suitable filter and modifies the values to suit the frequency band and impedance level.

4.6 Approximations for Low-Pass Filters

Before we go into the details of the approximations, we first pose a question: what kind of properties should a lowpass filter have to transfer, without distortion, a certain frequency band and suppress completely all other frequencies. The thought comes immediately to mind that all components should be amplified equally. This is true but not sufficient. In addition, the phase response must be a straight line; thus, the group delay must be a constant for all frequencies of interest. The conditions are sketched in Figure 4.15. We will see that this is impossible to achieve and that we have to accept some approximations when comparing with Figures 4.13(A) through 4.13(D) and Figure 4.14.

A lowpass filter, as indicated by the name, passes some frequency band starting from zero frequency and suppresses higher frequencies. One type of low-pass filter, called the polynomial filter, is described by transfer functions having a general form:

$$T(s) = K\frac{1}{\text{polynomial in } s}. \quad (4.30)$$

The polynomial must have all its roots (poles of the filter) in the left half of the complex plane to make the filter stable.

One such filter was suggested by Butterworth (Schaumann, 1990). He wanted to determine what the coefficients were of the polynomial in equation 4.30 to get a maximally flat amplitude low-pass transfer at $\omega = 0$. To make the problem unique, he also requested that the output of the filter at $\omega = 1$ should be 3 dB less than at $\omega = 0$.

We will not go into detail about how the polynomials were and are found; the steps are in any book on filters. Instead, we show the responses in Figure 4.16 for orders $n = 2$ to 10. We have used a logarithmic horizontal scale, and the responses start at $\omega = 0.1$. As we see, the approximation of a constant at low frequencies is reasonable for a high n, but the group delay

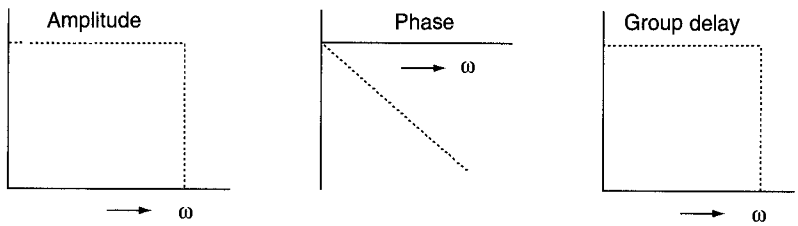
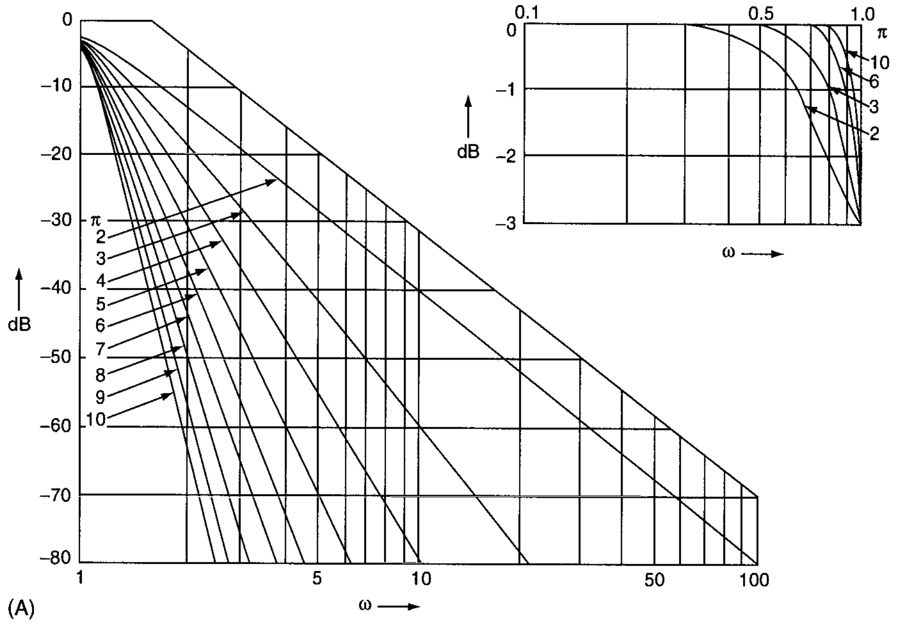
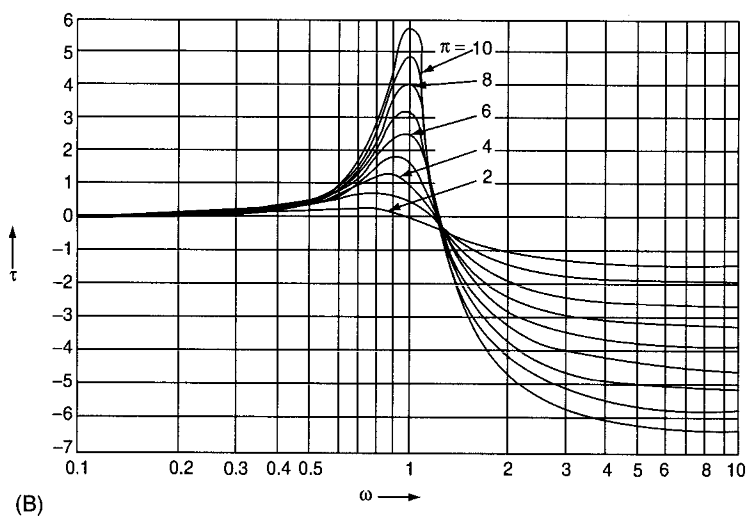


FIGURE 4.15 Ideal Responses for Amplitide, Phase, and Group Delay





π	τ(0)
2	1.414
3	1.388
4	2.613
5	3.236
6	3.863
7	4.493
8	5.125
9	5.758
10	6.392

FIGURE 4.16 (A) Selectivity Curves of Maximally Flat Filters (B) Group Delay of Maximally Flat Filters

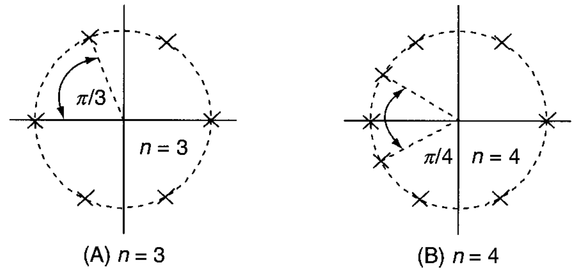


FIGURE 4.17 Poles' Positions for Maximally Flat (Butterworth) Filters, $n = 3$ and $n = 4$

turns out to have a peak at approximately $\omega = 1$, and the peak grows with the order n.

The poles of the normalized Butterworth (or maximally flat) filter lie on a circle with a radius $r = 1$. Figure 4.17 shows the situation for $n = 3$ and $n = 4$. Extension to higher degrees is easy to extrapolate from these two figures: all odd powers will have one real pole, and all even powers will have only complex conjugate poles. The poles of a few filters are in Table 4.1.

Another well-known types of polynomial filters include the Chebychev filters; they use special polynomials discovered by Russian scientist Chebychev. They are also described by the general Formula 4.30, but the requirements are different. In the passband, the amplitude response oscillates between two limits, like in Figure 4.18. The ripple is normally expressed in decibels. At $\omega = 1$, the response always drops by the amount of specified ripple, as in the figure. Compared with the maximally flat filters, Chebychev filters have better attenuation outside the passband and approximate a constant better in the frequency band from 0 to 1. Table 4.1 gives pole positions of Chebychev filters with 0.5 dB ripple.

If the requirements on the suppression of signals outside the passband are more severe, polynomial filters are not sufficient. Much better results are obtained with filters having transfer zeros placed on the imaginary axis. Because a pair of

TABLE 4.1 Poles of Filters

Order	Butterworth	Chebychev 0.5 dB ripple
2	$-0.7071068 \pm j0.7071068$	$-0.7128122 \pm j1.0040425$
3	-1.0000000	-0.6264565
	$-0.5000000 \pm j0.8660254$	$-0.3132282 \pm j1.0219275$
4	$-0.3826834 \pm j0.9238795$	$-0.1753531 \pm j1.0162529$
	$-0.9238795 \pm j0.3826834$	$-0.4233398 \pm j0.4209457$
5	-1.0000000	-0.3623196
	$-0.3090170 \pm j0.9510565$	$-0.1119629 \pm j1.0115574$
	$-0.8090170 \pm j0.5877852$	$-0.2931227 \pm j0.6251768$
6	$-0.2588190 \pm j0.9659258$	$-0.0776501 \pm j1.0084608$
	$-0.7071068 \pm j0.7071068$	$-0.2121440 \pm j0.7382446$
	$-0.9659258 \pm j0.2588190$	$-0.2897940 \pm j0.2702162$
7	-1.0000000	-0.2561700
	$-0.2225209 \pm j0.9749279$	$-0.0570032 \pm j1.0064085$
	$-0.6234898 \pm j0.7818315$	$-0.1597194 \pm j0.8070770$
	$-0.9009689 \pm j0.4338837$	$-0.2308012 \pm j0.4478939$

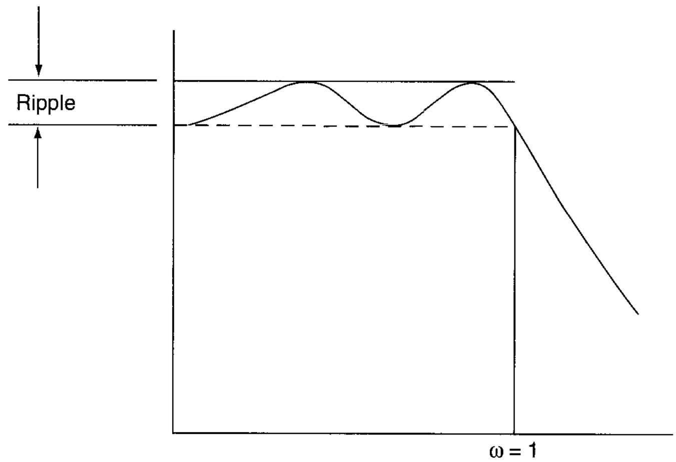


FIGURE 4.18 Amplitude Response of a Chebycher Filter

imaginary axis complex conjugate zeros is expressed by terms $(s + j\omega_i)(s - j\omega_i) = s^2 + \omega_i^2$, the transfer function will have the form:

$$T(s) = K\frac{\prod (s^2 + \omega_i^2)}{\text{polynomial in } s}. \tag{4.31}$$

A sketch of the pole-zero plot of some such lowpass filter was shown in Figure 4.10. As always, the roots of the polynomial must be in the left half of the plane. There are many possibilities for selecting the zeros and poles, all beyond the scope of this presentation. Fortunately, there exist numerous tables of filters in literature. Best known are the Cauer-parameter filters with equiripple responses in the passband and with equal minimal suppressions in the stopband. We have analyzed one such filter: see Figure 4.12 and its amplitude responses in Figures 4.13(A) through 4.13(D).

Before closing this section, we provide a general network for low-pass filters described by Formula 4.30. The elements are like those in Figure 4.19. Depending on the order of the filter, the last element before the load resistor will be either an inductor in series or a capacitor in parallel. We can have filters with different values of the resistors and also a design with only one resistor, R_L, but that does not change the general structure of the filter. Table 4.2 gives element values for normalized Butterworth filters with $R_E = R_L = 1$.

4.7 Transformations of Inductor Capacitor Low-Pass Filters

We spoke about low-pass filters and normalization previously because many normalized LC low-pass filters can be found in

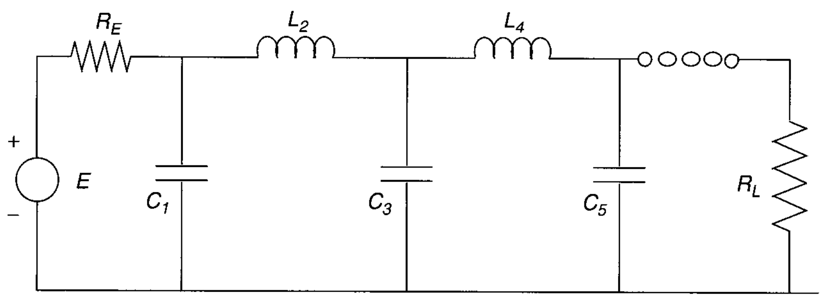


FIGURE 4.19 Realization of Polynomial Low-Pass Filters

TABLE 4.2 Elements Values of Butterworth Filters

Order	C_1	L_2	C_3	L_4	C_5	L_6	C_7
2	1.4142	1.4142					
3	1,0000	2,0000	1.0000				
4	0.7654	1.8478	1.8478	0,7654			
5	0.6180	1.6180	2,0000	1.6180	0.6180		
6	0.5176	1.4142	1.8319	1.8319	1.4142	0.5176	
7	0.4450	1,2470	1.8019	2.0000	1.8019	1.2470	0.4450

the literature, such as in Zverev (1976). One selects a suitable scaled filter and transforms it to the desired frequency and impedance level.

There are, however, additional possibilities for inductor capacitor (LC) low-pass filters. They can be transformed into band-pass, high-pass, or band-stop filters. We introduce these filters now.

4.7.1 Low-Pass into a High-Pass Filter

Consider the transformation:

$$z = \frac{\omega_0}{s}, \tag{4.32}$$

where z describes the complex frequency variable of the original scaled low-pass filter, possibly taken from tables, and s is the variable of the network to be realized. The meaning of ω_0 will become clear later. Multiply both sides of equation 4.32 by L. On the left is the impedance of an inductor. Simple arithmetic operations lead to:

$$zL = \frac{\omega_0 L}{s} = \frac{1}{s/\omega_0 L} = \frac{1}{sC'},$$

where on the right is an impedance of a capacitor with the value $C' = 1/\omega_0 L$. Considering multiplication of equation 4.32 by C, we get:

$$zC = \frac{\omega_0 C}{s} = \frac{1}{s/\omega_0 C} = \frac{1}{sL'}.$$

On the left is the admittance of a capacitor, and on the right is the admittance of an inductor: $L' = 1/\omega_0 C$. The transformation changes each capacitor into an inductor and vice versa; the filter is transformed into a high-pass.

We must still understand the meaning of ω_0. When evaluating the frequency domain responses, we always substitute $s = j\omega$. The frequency of interest is the cutoff frequency of the original low-pass filter, $z = j$. Inserting into equation 4.32, we get $-\omega_0 = \omega$. The sign only means that negative frequencies, existing in mathematics but not in reality, transform into positive frequencies and vice versa. The low-pass cutoff frequency of $\omega_s = 1$ transforms into the highpass cutoff frequency ω_0.

4.7.2 Low-Pass into a Band-Pass Filter

Consider next the transformation:

$$z = \frac{s}{\Delta} + \frac{\omega_0^2}{s\Delta} = \frac{s^2 + \omega_0^2}{s\Delta}. \tag{4.33}$$

In equation 4.33, z belongs to the original low-pass normalized filter. Multiply both sides by L to represent the impedance of an inductor in the z variable:

$$zL = \frac{sL}{\Delta} + \frac{\omega_0^2 L}{s\Delta} = \frac{sL}{\Delta} + \frac{1}{s\Delta/\omega_0^2 L} = sL' + \frac{1}{sC'}.$$

This means that the original inductor impedance was transformed into a series connection of two impedances. One of the elements is an inductor $L'_{ser} = L/\Delta$ and the other is a capacitor $C'_{ser} = \Delta/\omega_0^2 L$. Proceeding similarly for the admittance of a capacitor, we get:

$$zC = \frac{sC}{\Delta} + \frac{\omega_0^2 C}{s\Delta} = s\frac{C}{\Delta} + \frac{1}{s\Delta/\omega_0^2 C} = sC' + \frac{1}{sL'}.$$

The admittance of the capacitor is changed into the sum of two admittances, indicating a parallel connection. One of the elements is a capacitor $C'_{par} = C/\Delta$, and the other is an inductor

$$L'_{par} = \Delta/\omega_0^2 C.$$

To find additional properties of the transformation, we first form the product:

$$L'_{ser}C'_{ser} = L'_{par}C'_{par} = \frac{1}{\omega_0^2}.$$

This equation shows that the resonant frequencies of the parallel and series tuned circuits are the same, ω_0. Next, multiply equation 4.33 by the denominator to get:

$$s^2 - sz\Delta + \omega_0^2 = 0.$$

This is a quadratic equation with two solutions:

$$s_{1,2} = \frac{z\Delta}{2} \pm \sqrt{\frac{z^2\Delta^2}{4} - \omega_0^2}.$$

Now consider special points of the original filter. For $z = 0$, we get

$$s_{1,2}(z = 0) = \pm j\omega_0.$$

Zero frequency of the original filter is transformed into $\pm\omega_0$ point on the imaginary axis. Taking next the cutoff frequency of the low-pass filter, $z = \pm j$, we get four points:

$$s_{1,2,3,4} = \pm\frac{j\Delta}{2} \pm j\sqrt{\omega_0^2 + \frac{\Delta^2}{4}}.$$

Only two of these points will be on the positive imaginary axis:

$$j\omega_1 = j\frac{\Delta}{2} + j\sqrt{\omega_0^2 + \frac{\Delta^2}{4}}.$$

$$j\omega_2 = -j\frac{\Delta}{2} + j\sqrt{\omega_0^2 + \frac{\Delta^2}{4}}.$$

Their difference is the following:

$$\omega_1 - \omega_2 = \Delta. \tag{4.34}$$

Their product is:

$$\omega_1\omega_2 = \omega_0^2. \tag{4.35}$$

Thus, ω_0 is the (geometric) center of the two frequencies, and Δ is the passband of the transformed filter.

4.7.3 Low-Pass into a Band-Stop Filter

Consider a transformation similar to equation 4.33:

$$\frac{1}{z} = \frac{s}{\Delta} + \frac{\omega_0^2}{s\Delta} = \frac{s^2 + \omega_0^2}{s\Delta}. \tag{4.36}$$

It transforms a low-pass into a band-stop. All the above steps remain valid, and only the elements will be transformed differently. Divide both sides by L to get:

$$\frac{1}{zL} = \frac{s}{\Delta L} + \frac{\omega_0^2}{s\Delta L}.$$

On the left is an admittance, and on the right is the sum of two admittances, indicating that the elements will be in parallel: $C'_{par} = 1/\Delta L$ and $L'_{par} = \Delta L/\omega_0^2$. A similar division by C will result in:

$$\frac{1}{zC} = \frac{s}{\Delta C} + \frac{\omega_0^2}{s\Delta C}$$

On the left side is an impedance, and on the right side is the sum of two impedances, indicating that the elements will be in series $L'_{ser} = 1/\Delta C$ and $C'_{ser} = \Delta C/\omega_0^2$. All three transformations are summarized in Figure 4.20 for easy understanding. As an example, we will take the third order maximally flat filter from Table 4.2 and its realization in Figure 4.21(A). We then transform it into a band-pass filter with center frequency $\omega_0 = 1$ rad/s and with the bandwidth $\Delta = 0.1$ rad/s. The transformed band-pass filter is in Figure 4.21(B) and its amplitude response in Figure 4.22. Using equations from section 4.5, the band-pass filter can be transformed to any impedance level and any center frequency with a bandwidth equal to one tenth of the center frequency.

4.8 Realizability of Functions

Synthesis is a process in which we have a given function, and we try to find a network whose properties will be described by that function. In most cases, we try to do so for a voltage transfer function. The first question that comes into mind is whether any function can be realized as a network composed of passive elements: capacitors, inductors, resistors, and transformers. Rather obviously, the answer is no.

To make the problem treatable, we must start with a simple network function and establish conditions that the function must satisfy. The simplest network function is an impedance (or admittance) but since one is the inverse of the other, we can restrict ourselves to only one of them: the impedance. A large amount of work went into establishing the necessary and sufficient conditions for an impedance function composed of

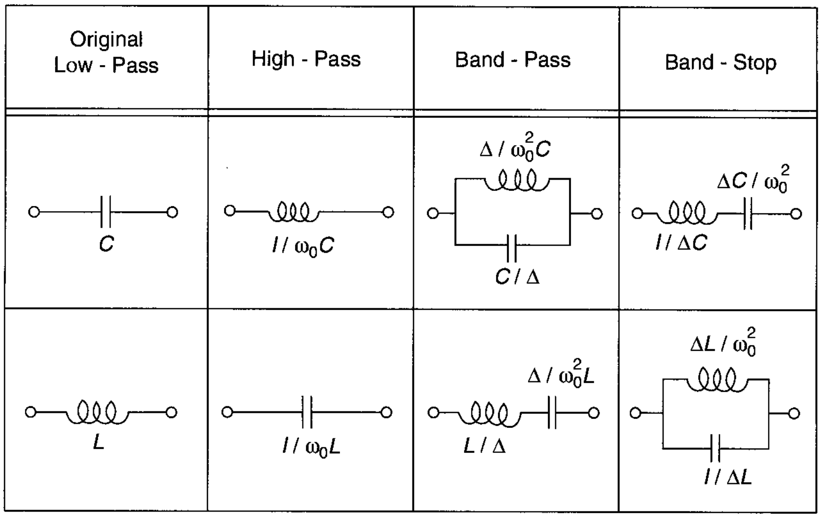


FIGURE 4.20 Transformations of Low-Pass Filters

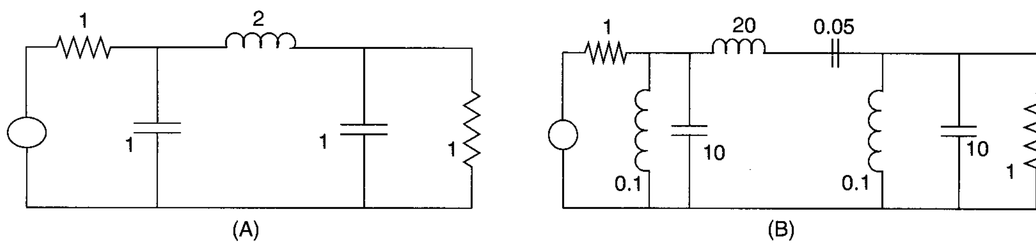


FIGURE 4.21 Transformation of a Butterworth Low-Pass Filter into a Band-Pass

L, *C*, and *R* elements, possibly with an ideal transformer. We will explain the conditions as a set of rules without trying to establish the reasons for these rules.

A rational function in the variable s can be realized as an impedance (admittance) if it satisfies the following rules:

1. The degree of the numerator and denominator may differ by at most one.
2. $Z(s)$ may not have any poles or zeros in the right half of the plane.
3. Poles and zeros in the left half of the plane may be multiple.
4. Poles on the imaginary axis must be simple and must have positive residues.
5. The function must satisfy the condition:

$$Re\,Z(s) \geq 0 \text{ if } Re\,s \geq 0. \tag{4.37}$$

The first condition is easy to establish by inspection. The second condition is a standard requirement for stability. Because the impedance and the admittance are the inverse of each other, the same must apply for the zeros. Stability is not destroyed by multiple poles in the left half plane, as stated in point 3. Point 4 would call for partial fraction expansion of the function. Unfortunately, all these steps are only necessary but not sufficient. The only necessary and sufficient condition is point 5, which is difficult to test.

All of this information may seem very discouraging, but, fortunately, synthesis of completely arbitrary impedances is almost never needed. In most cases, we need to realize LC impedances (admittances), and the rules are considerably simplified here. Again, without trying to provide a proof, let us state that any LC impedance can be realized in the form of the circuit in Figure 4.23(A) and any LC admittance in the form of the network in Figure 4.23(B).

Consider the network Figure 4.23(A). The tuned circuits have the impedance:

$$Z_i = \frac{1}{C_i} \frac{s}{s^2 + 1/L_i C_i}. \tag{4.38}$$

We usually define:

$$k_i = \frac{1}{2C_i}. \tag{4.39}$$

$$\omega_i^2 = \frac{1}{L_i C_i}. \tag{4.40}$$

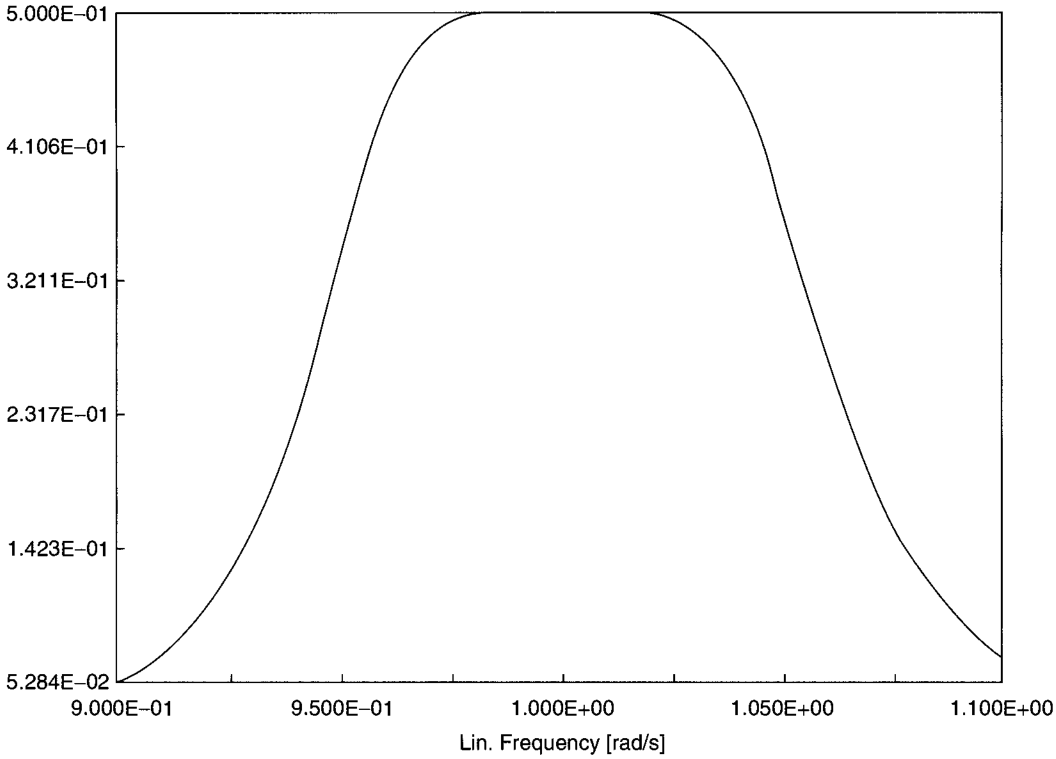


FIGURE 4.22 Amplitude Response of the Band-Pass Filter in Figure 4.21

The variable k_i represents the residues of the poles on the imaginary axis, $p_i = \pm j\omega_i$, and ω_i is the resonant frequency of the circuit. This gives us the possibility of writing a general LC impedance in the form:

$$Z_{LC} = sL + \frac{1}{sC} + \sum \frac{2k_i s}{s^2 + \omega_i^2}. \quad (4.41)$$

Any of the components in equation 4.41 may be missing. A similar expression could be derived for the network in Figure 4.23(B). We could now use a computer, plot various impedances, and study the results. We will summarize them for you:

1. Z_{LC} or Y_{LC} have only simple poles on the imaginary axis.
2. The residues of the poles are positive.
3. The zeros and poles on the imaginary axis must alternate.
4. Z_{LC} or Y_{LC} are expressed as a ratio of an even and an odd polynomial.
5. A pole or a zero may appear at the origin or at infinity, but the alternating nature of the poles and zeros must be preserved.

These results can be summarized by the simple sketches in Figure 4.24.

Synthesis of filters is usually based on the two-port theory. Using overall expressions for the whole filter, we separate the LC two-port from the loading resistors and apply synthesis to find the elements of the LC two-port. Again, as could be expected, some restrictions apply for the overall network, but the conditions are much more relaxed than for the impedances. A voltage or current transfer function can be realized as an LC two-port with loading resistors if the following statements are true:

1. The degree of the numerator, M, and denominator, N, satisfy $M \leq N$.
2. Zeros are usually on the imaginary axis, but theoretically they can be anywhere.
3. Poles must be in the left half of the plane.

Networks composed of passive elements always have $z_{12} = z_{21}$. In addition, z_{11} and z_{22} will have the same poles as $z_{12} = z_{21}$. Both z_{11} and z_{22} can have additional poles (called private poles), as we will show later. If the network components are only L and C, then z_{11} and z_{22} must satisfy the conditions on LC impedances discussed previously.

We now indicate the simplest method to extract the LC two-port from the network function. Consider the network in Figure 4.25. We wish to find the transfer impedance V_2/I_1. To do so, we need the second equation from the equation set 4.5,

$$V_2 = z_{21}I_1 + z_{22}I_2. \quad (4.42)$$

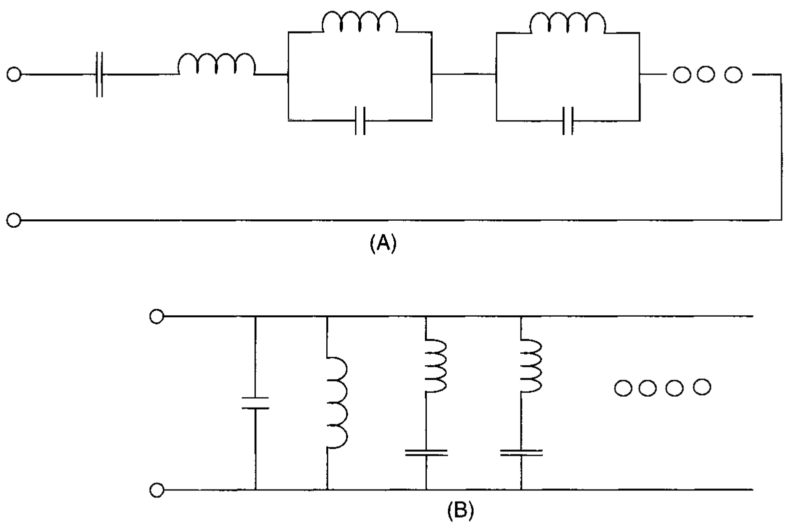


FIGURE 4.23 A General Form of an LC: (A) Impedance and (B) Admittance

After the indicated connection, we have:

$$-I_2 = \frac{V_2}{R}.$$

Inserting into equation 4.42 and eliminating I_2, we obtain:

$$Z_{TR} = \frac{V_2}{I_1} = \frac{z_{21}}{1 + z_{22}/R}. \tag{4.43}$$

As we explained in section 4.5, we can simplify the expression by normalizing the impedance level to $R = 1$. Now consider a function having the form:

$$f = \frac{g}{m + n}, \tag{4.44}$$

where m collects all the even terms of the denominator and n collects all the odd terms. All roots of the denominator must be in the left half of the plane. We can rewrite equation 4.44 in one of the two forms:

$$f = \frac{g/m}{1 + m/n}. \tag{4.45}$$

or

$$f = \frac{g/m}{1 + n/m}. \tag{4.46}$$

The ratios in the denominator, n/m or m/n, satisfy the conditions imposed on z_{22} of an LC network. The problem is reduced to the synthesis of z_{22}, taking into account the properties of z_{21}. We will describe such synthesis in section 4.10.

We have chosen this primitive case for demonstration only. Practical synthesis steps usually require voltage transfer with a loading resistor and with an input resistor. The steps that extract properties of the LC two-port are more complicated but are available in books focusing on synthesis of filters.

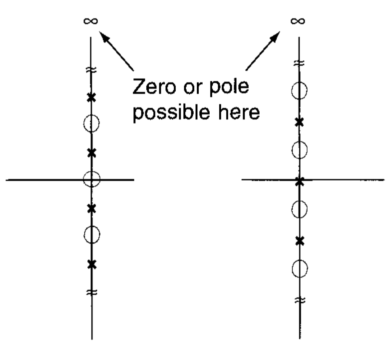


FIGURE 4.24 Possible Positions of LC Network Poles and Zeros

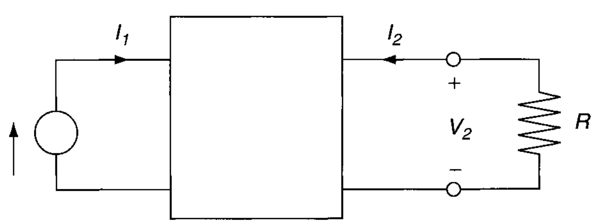


FIGURE 4.25 Deriving Z_{TR} for a Loaded Two-Port

4.9 Synthesis of LC One-Ports

For LC impedances or admittances, we stated the rules that the function is represented by the ratio of an even and odd polynomial and the poles and zeros must interchange, be simple, and have positive residues. To find the partial fractions, we must use a root-finding routine and actually get the zeros and poles first. To calculate the residues of the poles, we use the following formulas:

$$\text{Pole at the origin:}\quad k_0 = sZ(s)|_{s=0}. \tag{4.47}$$

$$\text{Pole at infinity:}\quad k_\infty = \frac{Z(s)}{s}|_{s=\infty}. \tag{4.48}$$

$$\text{Two poles at } \pm j\omega_i\text{: } 2k = \frac{s^2+\omega_i^2}{p} Z(s)|_{s^2=-\omega_i^2}. \tag{4.49}$$

In formulas 4.47 through 4.49, it is first necessary to cancel the additional terms against equal terms in $Z(s)$ and then substitute the indicated values for s. Suppose now that we consider an admittance in the form:

$$Y(s) = k_\infty s + \frac{k_0}{s} + \sum_{i=1}^{n} \frac{2k_i s}{s^2+\omega_i^2}. \tag{4.50}$$

The first term is $C = k_\infty$, and the second is $L = 1/k_0$. The remaining terms can be written as:

$$Y_i(s) = \frac{2k_i s}{s^2+\omega_i^2} = \frac{1}{s/2k_i + \omega_i^2/2k_i s}, \tag{4.51}$$

where

$$L_i = \frac{1}{2k_i}. \tag{4.52}$$

$$C_i = \frac{2k_i}{\omega_i^2}. \tag{4.53}$$

The admittance realized this way was shown in Figure 4.23(B). Should we consider the function in equation 4.50 as an impedance, similar steps would lead to the network in Figure 4.23(A). These networks have the name Foster canonical forms. Another method, originally due to Cauer, gives impedances or admittances in the form of a ladder. As an example, consider the impedance:

$$Z = \frac{s^4+10s^2+9}{s^3+4s} = \frac{(s^2+1)(s^2+9)}{s(s^2+4)}. \tag{4.54}$$

Subtract from equation 4.54 an impedance sL (a pole at infinity). This gives the following:

$$Z_2 = \frac{s^4+10s^2+9}{s^3+4s} - sL = \frac{s^4(1-L)+s^2(10-4L)+9}{s^3+4s}.$$

We could now select L to be anywhere between zero and one, but if we select $L = 1$, the function simplifies to:

$$Z_2 = \frac{6s^2+9}{s^3+4s}.$$

The subtraction means that we have realized the first element in series, $L = 1$, and removed a pole at infinity. The remaining function can now be inverted to:

$$Y_2 = \frac{s^3+4s}{6s^2+9}.$$

This equation has a pole at infinity. We can continue by subtracting from it the admittance of a capacitor:

$$Y_3 = Y_2 - sC = \frac{s^3(1-6C)+s(4-9C)}{6s^2+9}.$$

If we select $C = 1/6$, we remove again a pole at infinity and realize a capacitor in parallel, $C = 1/6$, with the result:

$$Y_3 = \frac{5s/2}{6s^2+9}.$$

Y_3 can be inverted again:

$$Z_3 = \frac{12s}{5} + \frac{18}{5s}.$$

The first element is an inductor $L = 12/5$, the second a capacitor $C = 5/18$, both connected in series. The resulting network is in Figure 4.26.

The above expansion started from the highest powers. There exists another possibility by starting from the lowest power. Rewrite the impedance from 4.54 in the form:

$$Z = \frac{9+10s^2+s^4}{4s+s^3}. \tag{4.55}$$

and subtract $1/sC$, a pole at the origin:

$$Z_2 = Z - \frac{1}{sC} = \frac{(9-4/C)+s^2(10-1/C)+s^4}{4s+s^3}.$$

If we select $C = 4/9$ as a capacitor in series, the first term disappears, and we remove the pole at $s = 0$ to get:

$$Z_2 = \frac{31s/4+s^3}{4+s^2}.$$

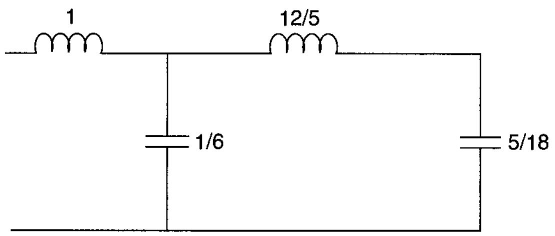


FIGURE 4.26 Synthesis of Equation 4.54 by Extracting Poles at Infinity

The remainder can be inverted and the admittance of an inductor subtracted again:

$$Y_3 = Y_2 - \frac{1}{sL} = \frac{(4 - 31/4L) + s^2(1 - 1/L)}{\frac{31}{4}s + s^3}.$$

The choice $L = 31/16$, connected in parallel, removes the first term and another pole at the origin. The process can be continued, with the resulting network shown in Figure 4.27. It is not necessary to continue one type of the expansion to the end. It is always possible, at any step, to rewrite the remaining function like we did going from equations 4.54 to 4.55 and continue. In addition, we need not remove one of the terms completely. All this indicates that synthesis of LC impedances or admittances is far from a unique procedure.

Let us return to equation 4.54, where we subtracted $L = 1$; we subtract now only $L = 0.5$. The result will be as follows:

$$Z_2 = \frac{0.5s^4 + 8s^2 + 9}{s^3 + 4s} = \frac{0.5(s^2 + 1.22)(s^2 + 14.78)}{s^3 + 4s}. \quad (4.56)$$

We see that the partial removal did not simplify the function but shifted zeros of the original function into new positions. Normally, if no other conditions are imposed, we would always remove the full value because then the resulting number of elements is minimal. Partial removal is used in the synthesis of two-port networks, as is shown later in this chapter.

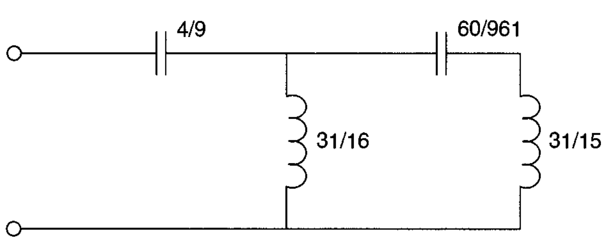


FIGURE 4.27 Synthesis of Equation 4.55 by Extracting Poles at Zero

4.10 Synthesis of LC Two-Port Networks

4.10.1 Transfer Zeros at Infinity

In this section, we indicate synthesis steps of two-ports. It is a fairly complicated procedure, and we will try to explain directly with examples.

An LC two-port is described by its impedance or admittance parameters, equations 4.5 or 4.7. We will use the impedance parameters and assume that the functions have been decomposed into partial fractions. Because for passive networks $z_{12} = z_{21}$, we need (for full description only):

$$z_{11} = \frac{k_0^{(11)}}{s} + \sum \frac{2k_i^{(11)}s}{s^2 + \omega_i^2} + k_\infty^{(11)}s$$

possibly plus some LC impedance.

$$z_{22} = \frac{k_0^{(22)}}{s} + \sum \frac{2k_i^{(22)}s}{s^2 + \omega_i^2} + k_\infty^{(22)}s$$

possibly plus some LC impedance.

$$z_{12}(= z_{21}) = \frac{k_0^{(12)}}{s} + \sum \frac{2k_i^{(12)}s}{s^2 + \omega_i^2} + k_\infty^{(12)}s.$$

The words *possibly in addition LC impedance* indicate that additional terms may exist in z_{11} and/or z_{22}. These are called private impedances and do not influence z_{12}. They would appear as in Figure 4.28(A) for impedance parameters and as in Figure 4.28(B) for admittance parameters. After their removal, all z_{ij} must have the same poles, the residues of z_{11} and z_{22} must be positive, and residues of z_{12} may be positive or negative.

Before we start with the synthesis example, let us state that a removal of an element, which is supposed to influence z_{12} (or y_{12}), must be connected as in Figure 4.29(A) or 4.29(B). Let us now have the following two functions:

$$z_{11} = s + \frac{9}{4s} + \frac{15}{4}\frac{s}{s^2 + 4}.$$
$$z_{12} = \frac{1}{4s} + \frac{-s/4}{s^2 + 4} = \frac{1}{s(s^2 + 4)}. \quad (4.57)$$

The first term in z_{11} is a private impedance (the pole at infinity does not appear in z_{12}) and can be removed as an inductor $L = 1$ in series. The remaining functions are these:

$$z_{11}^* = Z_1 = \frac{9}{4s} + \frac{15}{4}\frac{s}{s^2 + 4} = \frac{6s^2 + 9}{s(s^2 + 4)}.$$

$$z_{12} = \frac{1}{s(s^2 + 4)}.$$

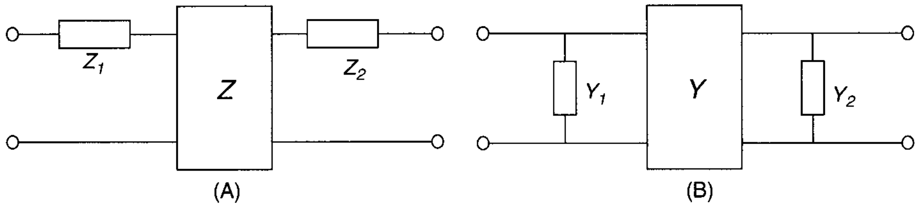


FIGURE 4.28 Removing Private (A) Impedances and (B) Admittances

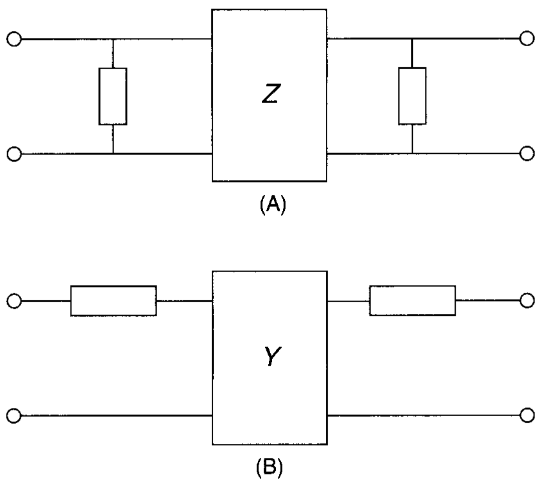


FIGURE 4.29 Removals Affecting Transfer Immittances and Connection of the Last Element for (A) Impedance Description and (B) Admittance Description

The poles of both functions are the same, z_{12} has only a constant in the numerator, and three powers of s are in the denominator. If we insert infinity for any s in the denominator, the function will become zero, and we say that z_{12} has three zeros at infinity. As was shown in the previous section, we cannot remove zeros as elements, but we can remove them as poles of inverted functions. Subtracting the admittance sC from $Y_1 = 1/Z_1$ leads to:

$$Y_2 = \frac{s^3 + 4s}{6s^2 + 9} - sC = \frac{s^3(1 - 6C) + s(4 - 9C)}{6s^2 + 9}$$

Selecting $C = 1/6$ (realized as a capacitor in parallel) reduces the admittance to

$$Y_2 = \frac{5s}{12s^2 + 18}.$$

and takes care of one of the transfer zeros of z_{12}. The remaining function, Y_2, has a zero at infinity. We can invert and remove a pole at infinity from:

$$Z_3 = \frac{1}{Y_2} - sL = \frac{s^3(12 - 5L) + 18}{5s}.$$

We realize $L = 12/5$ as an inductor in series. This has taken care of the second transfer zero of z_{12}. The remaining function is $Z_4 = 18/5\,s$. By inverting it, we have $Y_4 = 5\,s/18$ and remove the last transfer zero as a pole at infinity by connecting a capacitor $C = 5/18$ in parallel. The whole network is in Figure 4.30. Let us now check our result logically. Inductors in series obstruct transfer of high frequencies and capacitors in parallel represent short circuits at high frequencies. Hence, each of the capacitors in parallel and the middle inductor in series will indeed help in suppressing high frequencies and create together three zeros at infinity.

We have used the above more complicated procedure to prepare the reader for synthesis of two-ports with finite transfer zeros. We could have achieved the same by dividing the numerator by the denominator in each of the steps. In fact, if the reader knows expansions into continued fractions, he or she should try it on $1/Z_1$. It is the fastest way for the removal of transfer zeros at infinity.

4.10.2 Transfer Zeros on the Imaginary Axis

As we have demonstrated in equation 4.56, partial removal does not reduce the degree of the function but shifts the zeros to different places. This feature is used in the synthesis. We will indicate the method in an example. Consider:

$$z_{11} = \frac{(s^2+2)(s^2+6)}{s(s^2+3)} = \frac{s^4 + 8s^2 + 12}{s(s^2+3)} = s + \frac{4}{s} + \frac{s}{s^2+3}.$$
$$z_{12} = \frac{(s^2+1)(s^2+6)}{s(s^2+3)} = s + \frac{4}{3s} + \frac{2s}{3(s^2+3)}. \quad (4.58)$$

The transfer impedance has transfer zeros at $s = \pm j1$ and $s = \pm j2$. Both z_{11} and z_{12} have the same poles. There are no private poles in z_{11}; the functions are ratios of even and odd polynomials, zeros, and poles of z_{11} interchange. Thus, all the conditions for realizability are satisfied.

In the first step, we wish to take into consideration the transfer zero at $j1$. We substitute this value into z_{11} to obtain:

$$z_{11}(j1) = \frac{5}{2j}. \quad (4.59)$$

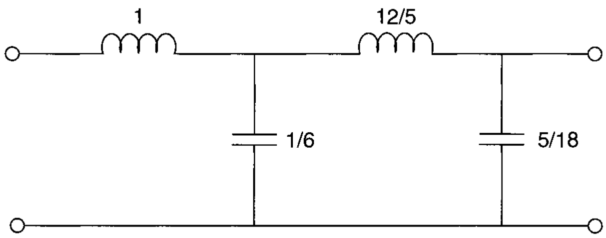


FIGURE 4.30 Synthesis of Equation 4.57

Equation 4.59 behaves as a capacitance; at $\omega_1 = 1$, we have $5/2j = 1/j\omega_1 C = 1/j1C$ from which $C = 2/5$. The capacitance and z_{11} have a pole at the origin, and we subtract:

$$Z_2 = z_{11} - \frac{5}{2s} = \frac{s^4 + 11s^2/2 + 9/2}{s(s^2+3)}. \quad (4.60)$$

The intention is to get the numerator polynomial with zeros at $s = \pm j1$. We can now divide the numerator of equation 4.60 by the term $s^2 + 1$ and obtain the decomposition:

$$Z_2 = \frac{(s^2+1)(s^2+9/2)}{s(s^2+3)}. \quad (4.61)$$

Notice the neat trick we used to shift the zeros by first evaluating equation 4.59. Using the above steps, we have realized the left capacitor in Figure 4.31. Z_2 now has the same number of zeros as the transfer function. We remove them as poles of the inverted function:

$$Y_2 = \frac{s(s^2+3)}{(s^2+1)(s^2+9/2)}. \quad (4.62)$$

To be removed is an admittance of a series-tuned circuit of the form $2k_i s/(s^2 + \omega_i^2)$. This is done by the formula 4.49:

$$2k_1 = \frac{(s^2+1)}{s}\frac{s(s^2+3)}{(s^2+1)(s^2+9/2)}\Big|_{s^2=-1} = \frac{4}{7}. \quad (4.63)$$

The element values of the tuned circuit are:

$$Y_{tc} = \frac{4s/7}{s^2+1} = \frac{1}{7s/4 + 7/4s},$$

with the result $L = 7/4$ and $C = 4/7$. The series tuned circuit, connected in parallel, is on the left of Figure 4.31. In the next step, we remove the expression for the tuned circuit from Y_2:

$$Y_3 = \frac{s(s^2+3)}{(s^2+1)(s^2+9/2)} - \frac{4s/7}{s^2+1} = \frac{3s}{7(s^2+9/2)}. \quad (4.64)$$

Then we return to the impedance:

$$Z_3 = \frac{7}{3}\frac{s^2+9/2}{s}.$$

There is another transfer zero to be removed, $s = j2$. The procedure is repeated by first evaluating $Z_3(j2) = 7/j12$. It behaves as a capacitance equal to $7/j12 = 7/j\omega_2 C_2 = 7/j2C_2$ with the result $C_2 = 6/7$. The impedance corresponding to the capacitance has a pole at the origin, and Z_3 has the same pole, so a partial removal of the pole is possible:

$$Z_4 = Z_3 - \frac{1}{sC_2} = \frac{7(s^2+9/2)}{3s} - \frac{7}{6s} = \frac{7(s^2+4)}{3s}.$$

Because now Z_4 has a zero at the proper place, we can remove it as a pole of a tuned circuit:

$$Y_4 = \frac{3s}{7s^2+28},$$

with the result $L = 7/3$ and $C = 3/28$, see Figure 4.31.

4.11 All-Pass Networks

In filter design, we place the transfer zeros almost always on the imaginary axis because this secures largest suppression of the signal in the stopband. In most cases, a low-pass filter should have approximately constant transfer of low frequencies and rapid suppression of frequencies beyond the specified passband. This may result in almost constant transfer of the desired frequencies, but the group delay response is then far from ideal. We have analyzed one such filter, Figure 4.12, with its amplitude responses in Figure 4.13(A) through Figure 4.13D and with the group delay in Figure 4.14.

In special cases, it may be necessary to compensate for the nonideal group delay response, and this can be done with all-pass networks. They modify only the group delay and pass all frequencies without attenuation.

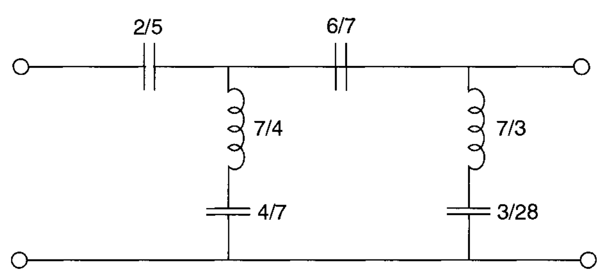


FIGURE 4.31 Synthesis of Equation 4.58

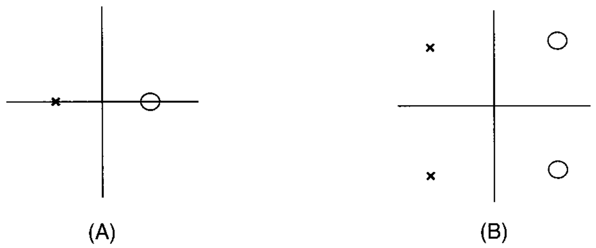


FIGURE 4.32 Possible Pole-Zero Positions for All-Pass Networks

Consider the symmetrical pole and zero positions sketched in Figure 4.32(A) and Figure 4.32(B) and the formulas for amplitude and group delay responses, equations 4.25 and 4.27. Symmetrical positions of both axes make $\alpha_i = -\gamma_i$ and $\beta_i = \delta_i$. When these are inserted into equation 4.25, the numerator cancels against the denominator and the expression in large brackets becomes equal to one. Inserted into equation 4.27, the two terms add and:

$$\tau_{\text{all-pass}} = 2\sum_{i=1}^{M} \frac{\alpha_i}{\alpha_i^2 + (\omega - \beta_i)^2}. \tag{4.65}$$

For further study, we will use the two-port theory and the impedance parameters. Consider the network in Figure 4.33, and remove the dotted resistor. Without it, we have two voltage dividers:

$$V_2 = V_B - V_A = \frac{Z_1 - Z_2}{Z_1 + Z_2} V_1.$$

The current flowing from the source into the two branches will be:

$$I_1 = \frac{2V_1}{Z_1 + Z_2}$$

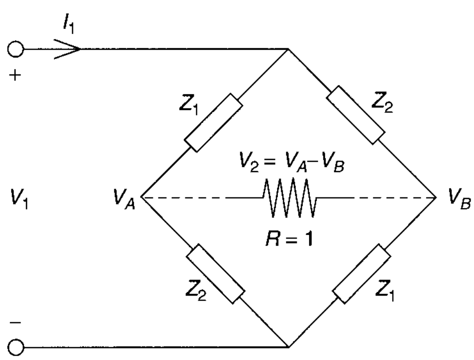


FIGURE 4.33 Deriving Properties of an All-Pass Network

and the entries of the two-port impedance matrix are:

$$Z = \begin{bmatrix} \dfrac{Z_2 + Z_1}{2} & \dfrac{Z_2 - Z_1}{2} \\ \dfrac{Z_2 - Z_1}{2} & \dfrac{Z_2 + Z_1}{2} \end{bmatrix} \tag{4.66}$$

If we connect the dotted resistor, the situation changes:

$$Z_{TR} = \frac{z_{21}}{1 + z_{22}}, \tag{4.67}$$

derived already in equation 4.43, and the input impedance becomes:

$$Z_{in} = \frac{z_{11} + z_{11}z_{22} - z_{12}z_{21}}{1 + z_{22}}, \tag{4.68}$$

obtained in equation 4.10. Let us now impose the condition:

$$Z_1 Z_2 = 1. \tag{4.69}$$

It changes the matrix of equation 4.66 into:

$$Z = \begin{bmatrix} \dfrac{1 + Z_1^2}{2Z_1} & \dfrac{1 - Z_1^2}{2Z_1} \\ \dfrac{1 - Z_1^2}{2Z_1} & \dfrac{1 + Z_1^2}{2Z_1} \end{bmatrix} \tag{4.70}$$

Inserting these entries into equation 4.68, we get, after a few simple algebraic steps, the surprising result:

$$Z_{in} = 1. \tag{4.71}$$

The input impedance of the combination is equal to the loading resistor, irrespective of what the value of Z_1, as long as we maintain the condition that $Z_2 = 1/Z_1$. Inserting this condition into equation 4.67 results in:

$$Z_{TR} = \frac{1 - Z_1}{1 + Z_1}. \tag{4.72}$$

Let us now take the special case $Z_1 = sL$. The transfer impedance becomes $Z_{TR} = (1 - sL)/(1 + sL)$, and we get the design parameters shown in Figure 4.34. The figure uses an accepted practice to draw only two of the impedances and to indicate the presence of the other two by dashed lines.

Suppose we take for Z_1 a parallel tuned circuit with elements L and C. In such a case,

$$Z_1 = \frac{sL}{s^2LC + 1}. \tag{4.73}$$

Inserting into equation 4.72, we get:

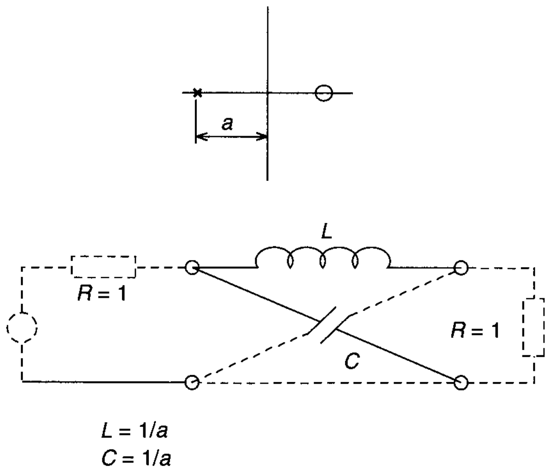


FIGURE 4.34 Position of One All-Pass Pole-Zero and Network Realization

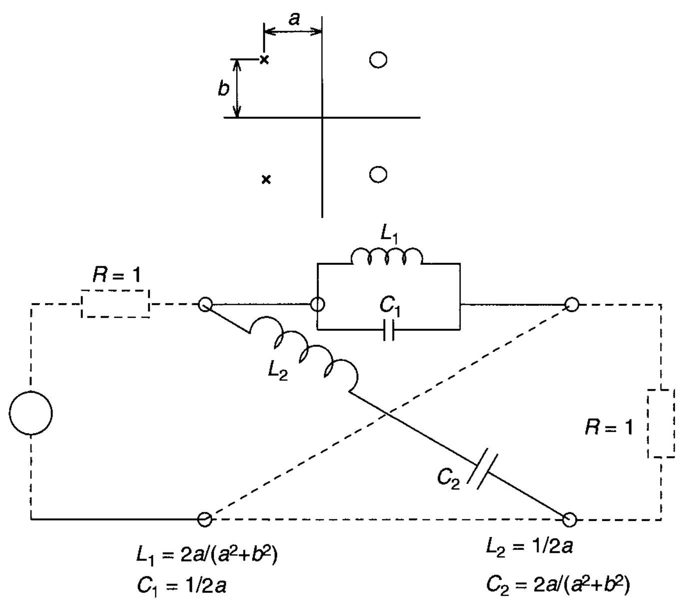


FIGURE 4.35 Positions of Two All-Pass Pole-Zero and Network Realization

$$Z_{TR} = \frac{s^2 - s/C + 1/LC}{s^2 + s/C + 1/LC}. \tag{4.74}$$

Comparing the denominator with

$$(s + a + jb)(s + a - jb) = s^2 + 2as + (a^2 + b^2),$$

we conclude that $C = 1/2a$ and $L = 2a/(a^2 + b^2)$. The network and its design values are in Figure 4.35. Remember that the loading resistor was normalized to unit value, and for other values of R, we must also scale impedances of these elements.

The above networks have one considerable disadvantage: the output is taken between two nodes and not with respect to ground. Connection to filters would have to be done through a transformer. All-pass networks with input and output taken with respect to ground also exist but require a transformer in the all-pass.

4.12 Summary

We have presented simplified steps for the synthesis of LC filters. The intention was to provide enough information for a general understanding and for encouraging the reading of more advanced books. The material should be sufficient for practical situations when tables of filters can be used. We have intentionally skipped synthesis of RC networks. The theory is available but is almost never used for designs, and is in many respects similar to what has been presented. Synthesis of RC networks can be found in specialized books. All references except Mitra (1969), which is an early text concerning active network synthesis, are classic texts about passive network synthesis.

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