5

Nonlinear Circuits

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5.1 Introduction

Nonlinearity plays a crucial role in many systems and qualitatively affects their behavior. For example, most circuits emanating from the designs of integrated electronic circuits and systems are nonlinear. It is not an overstatement to say that the majority of interesting electronic circuits and systems are nonlinear.

Circuit nonlinearity is often a desirable design feature. Many electronic circuits are designed to employ the nonlinear behavior of their components. The nonlinearity of the circuits' elements is exploited to provide functionality that could not be achieved with linear circuit elements. For example, nonlinear circuit elements are essential building blocks in many well-known electronic circuits, such as bistable circuits, flip-flops, static shift registers, static RAM cells, latch circuits, oscillators, and Schmitt triggers: they all require nonlinear elements to function properly. The global behavior of these circuits differs from the behavior of amplifiers or logic gates in a fundamental way: they must possess multiple, yet isolated, direct current (dc) operating points (also called equilibrium points). This is possible only if a nonlinear element (such as a transistor) is employed.

A **nonlinear circuit** or a **network** (a circuit with a relatively large number of components) consists of at least one nonlinear element, not counting the voltage and current independent sources. A circuit element is called nonlinear if its constitutive relationship between its voltage (established across) and its current (flowing through) is a nonlinear function or a nonlinear relation. All physical circuits are nonlinear. When analyzing circuits, we often simplify the behavior of circuit elements and model them as linear elements. This simplifies the model of the circuit to a linear system, which is then described by a set of linear equations that are easier to solve. In many cases, this simplification is not possible and the nonlinear elements govern the behavior of the circuit. In such cases, approximation to a linear circuit is possible only over a limited range of circuit variables (voltages and currents), over a limited range of circuit parameters (resistance, capacitance, or inductance), or over a limited range of environmental conditions (temperature, humidity, or aging) that may affect the behavior of the circuit. If global behavior of a circuit is sought, linearization is often not possible and the designer has to deal with the circuit's nonlinear behavior (Chua *et al.*, 1987; Hasler and Neirynck, 1986; Mathis, 1987; Willson, 1975).

As in the case of a **linear circuit**, Kirchhoff's voltage and current laws and Telegen's theorem are valid for nonlinear circuits. They deal only with a circuit's topology (the manner in which the circuit elements are connected together). Kirchhoff's voltage and current laws express linear relationships between a circuit's voltages or currents. The nonlinear relationships between circuit variables stem from the elements' constitutive relationships. As a result, the equations governing the behavior of a nonlinear circuit are nonlinear. In cases where a direct current response of the circuit is sought, the governing equations are nonlinear algebraic equations. In cases where transient behavior of a circuit is examined, the governing equations are nonlinear differential-algebraic equations.

Analysis of nonlinear circuits is more difficult than the analysis of linear circuits. Various tools have been used to understand and capture nonlinear circuit behavior. Some approaches employ quantitative (numerical) techniques and the use of circuit simulators for finding the distribution of a circuit's currents and voltages for a variety of waveforms supplied by sources. Other tools employ qualitative analyses methods, such as those dealing with the theory for establishing the number of dc operating points a circuit may possess or dealing with the analysis of stability of a circuit's operating point.

5.2 Models of Physical Circuit Elements

Circuit elements are models of physical components. Every physical component is nonlinear. The models that capture behavior of these components over a wide range of values of voltages and currents are nonlinear. To some degree of simplification, circuit elements may be modeled as linear elements that allow for an easier analysis and prediction of a circuit's behavior. These linear models are convenient simplifications that are often valid only over a limited range of the element's voltages and currents.

Another simplification that is often used when seeking to calculate the distribution of circuit's voltages and currents, is a piecewise-linear approximation of the nonlinear element's characteristics. The piecewise-linear approximation represents a nonlinear function with a set of nonoverlapping linear segments closely resembling the nonlinear characteristic. For example, this approach is used when calculating circuit voltages and currents for rather simple nonlinear electronic circuits or when applying certain computer-aided software design tools.

Nonlinear circuit elements may be classified based on their constitutive relationships (resistive and dynamical) and based on the number of the element's terminals.

5.2.1 Two-Terminal Nonlinear Resistive Elements

The most common two-terminal nonlinear element is a **nonlinear resistor**. Its constitutive relationship is:

$$f(i, v) = 0, \tag{5.1}$$

where f is a nonlinear mapping. In special cases, the mapping can be expressed as:

$$i = g(v), \tag{5.2}$$

when it is called a voltage-controlled resistor or:

$$v = h(i), \tag{5.3}$$

when it is a current-controlled resistor.

Examples of nonlinear characteristics of nonlinear two-terminal resistors are shown in Figure 5.1(A). Nonlinear resistors commonly used in electronic circuits are diodes: exponential, zener, and tunnel (Esaki) diodes. They are built from semiconductor materials and are often used in the design of electronic circuits along with linear elements that provide appropriate biasing for the nonlinear components. A circuit symbol for an exponential diode and its characteristic are given in Figure 5.1(B). The diode model is often simplified and the current flowing through the diode in the **reverse biased region** is assumed to be zero. Further simplification leads to a model of a switch: the diode is either *on* (voltage across the diode terminals is zero) or *off* (current flowing through the diode is zero). These simplified diode models are also nonlinear. Three such characteristics are shown in Figure 5.1(B). The zener diode, shown in Figure 5.1(C), is another commonly used electrical component. Tunnel diodes have more complex nonlinear characteristic, as shown in Figure 5.1(D). This nonlinear element is qualitatively different from exponential diodes. It permits multiple values of voltages across the diode for the same value of current flowing through it. Its characteristic has a region where its derivative (differential resistance) is negative. Silicon-controlled rectifier, used in many switching circuits, is a two-terminal nonlinear element with a similar behavior. It is a current-controlled resistor that permits multiple values of currents flowing through it for the same value of voltage across its terminals. A symbol for the silicon-controlled rectifier and its current-controlled characteristic are shown in Figure 5.1(E).

Two simple nonlinear elements, called **nullator** and **norator**, have been introduced mainly for theoretical analysis of nonlinear circuits. The nullator is a two-terminal element defined by the constitutive relations:

$$v = i = 0. \tag{5.4}$$

The current and voltage of a norator are not subject to any constraints. Combinations of nullators and norators have been used to model multiterminal nonlinear elements, such as transistors and operational amplifiers.

Voltage- and current-independent sources are also considered to belong to the family of nonlinear resistors. In contrast to the described nonlinear resistors, which are all passive for the choice of their characteristics presented here, voltage and current independent sources are active circuit elements.

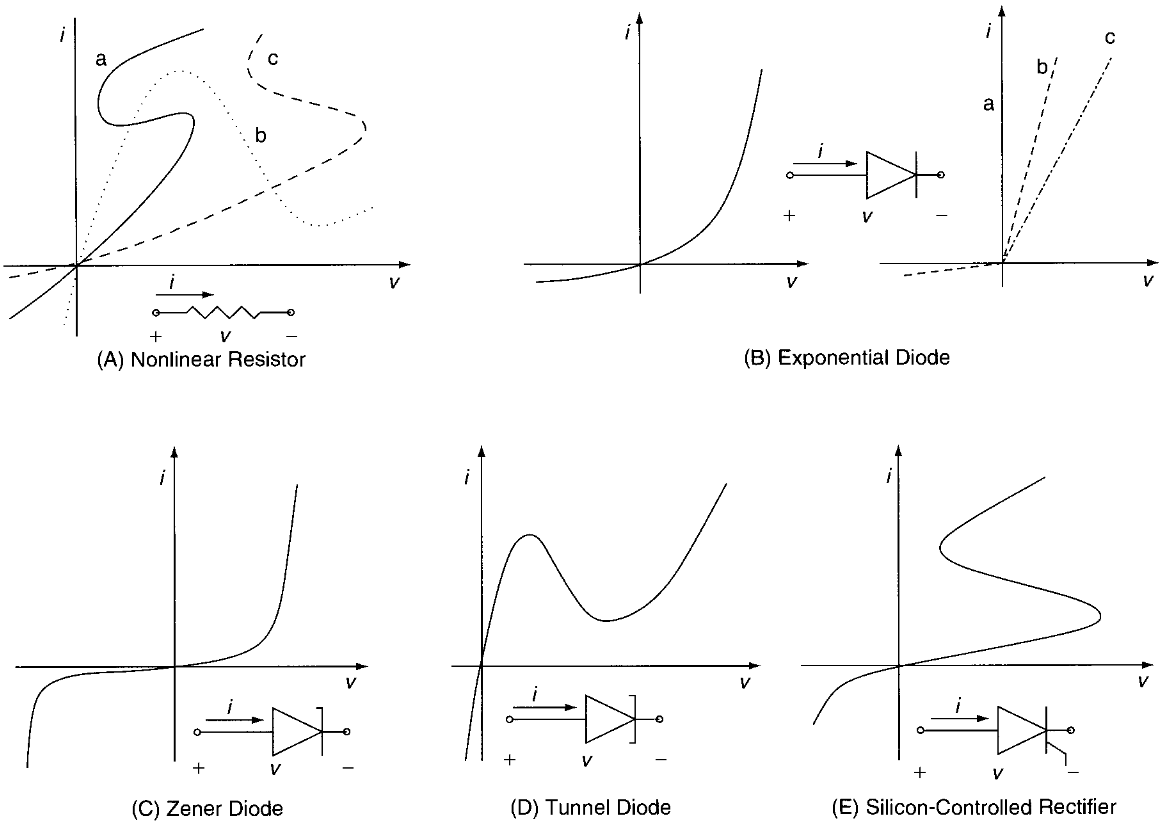

FIGURE 5.1 Two-Terminal Nonlinear Circuit Elements: (A) Nonlinear resistor and examples of three nonlinear constitutive characteristics (a. general, b. voltage-controlled, and c. current-controlled). (B) Exponential diode and its three simplified characteristics. (C) Zener diode and its characteristic with a breakdown region for negative values of the diode voltage. (D) Tunnel diode and its voltage-controlled characteristic. (E) Silicon-controlled rectifier and its current-controlled characteristic.

5.2.2 Two-Terminal Nonlinear Dynamical Elements

Nonlinear resistors have a common characteristic that their constitutive relationships are described by nonlinear algebraic equations. **Dynamical nonlinear elements**, such as nonlinear inductors and capacitors, are described by nonlinear differential equations. The constitutive relation of a nonlinear capacitor is:

$$f(v,\ q) = 0, \tag{5.5}$$

where the charge q and the current i are related by:

$$i = \frac{dq}{dt}. \tag{5.6}$$

The constitutive relation for a nonlinear inductor is:

$$f(\phi,\ i) = 0, \tag{5.7}$$

where the flux ϕ and the voltage v are related by:

$$v = \frac{d\phi}{dt}. \tag{5.8}$$

A generalization of the nonlinear resistor that has memory, called a **memristor**, is a one-port element defined by a constitutive relationship:

$$f(\phi, q) = 0, \tag{5.9}$$

where the flux ϕ and charge q are defined in the usual sense. In special cases, the element may model a nonlinear resistor, capacitor, or inductor.

5.2.3 Three-Terminal Nonlinear Resistive Elements

A three-terminal nonlinear circuit element commonly used in the design of electronic circuits is a bipolar junction transistor (Ebers and Moll, 1954; Getreu, 1976). It is a current-controlled nonlinear element used in the design of various integrated circuits. It is used in circuit designs that

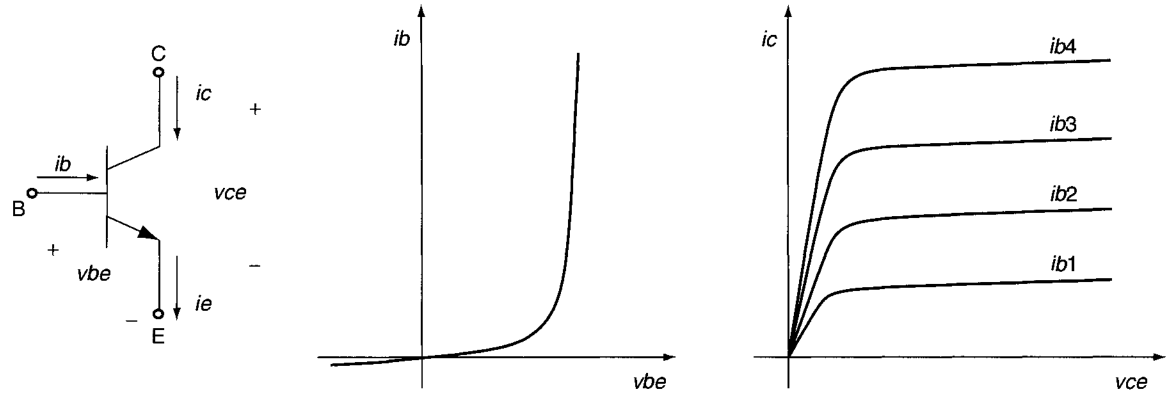

FIGURE 5.2 Three-Terminal Nonlinear Resistive Element: A Circuit Symbol and Characteristics of an *n–p–n* Bipolar Junction Transistor

rely on the linear region of the element's characteristic (logic gates and arithmetic circuits) to designs that rely on the element's nonlinear behavior (flip-flops, static memory cells, and Schmitt triggers). A circuit symbol of the bipolar junction transistor and its simplified characteristics are shown in Figure 5.2. The large-signal behavior of a bipolar junction transistor is usually specified graphically by a pair of graphs with families of curves. A variety of field-effect transistors (the most popular being a metal-oxide semiconductor) are voltage-controlled nonlinear elements used in a similar design fashion as the bipolar junction transistors (Massobrio and Antognetti, 1993).

5.2.4 Multiterminal Nonlinear Resistive Elements

A commonly used multiterminal circuit element is an **operational amplifier** (op-amp). It is usually built from linear resistors and bipolar junction or filed-effect transistors. The behavior of an operational amplifier is often simplified and its circuit is modeled as a linear multiterminal element. Over the limited range of the op-amp input voltage, the relationship between the op-amp input and output voltages is assumed to be linear. The nonlinearity of the op-amp circuit elements is employed in such a way that the behavior of the op-amp can be approximated with a simple linear characteristic. Nevertheless, even in the case of such simplification, the saturation of the op-amp characteristic for larger values of input voltages needs to be taken into account. This makes the op-amp a nonlinear element. The op-amp circuit symbol and its simplified characteristic are shown in Figure 5.3.

5.2.5 Qualitative Properties of Circuit Elements

Two simple albeit fundamental attributes of circuit elements are **passivity** and **no-gain**. They have both proven useful in applying certain modern mathematical techniques for analyzing and solving nonlinear circuit equations. A circuit element is passive if it absorbs power (i.e., if at any operating point, the net power delivered to the element is nonnegative). A nonlinear resistor is passive if for any pair (v, i) of voltage and current, its constitutive relationship satisfies:

$$vi \geq 0. \tag{10}$$

Otherwise, the element is active. Hence, a nonlinear resistor is passive if its characteristic lies in the first and third quadrant. A circuit is passive if all its elements are passive.

A multiterminal element is a **no-gain** element if every connected circuit containing that element, positive resistors, and independent sources possesses the no-gain property. A circuit possesses the no-gain property if, for each solution to the dc network equations, the magnitude of the voltage between any pair of nodes is not greater than the sum of the magnitudes of the voltages appearing across the network's independent sources. In addition, the magnitude of the current flowing through any element of the network must not be greater than the sum of the magnitudes of the currents flowing through the network's independent sources. For example, when considering a multiterminal's large-signal dc behavior, Ebers-Moll modeled bipolar junction transistors, filed-effect transistors, silicon-controlled rectifiers, and many other three-terminal devices are passive and incapable of producing voltage or current gains (Gopinath and Mitra, 1971; Willson, 1975).

Clearly, the no-gain criterion is more restrictive than passive. That is, passivity always follows as a consequence of no-gain, although it is quite possible to cite examples (e.g., the ideal transformer) of passive components capable of providing voltage or current gains.

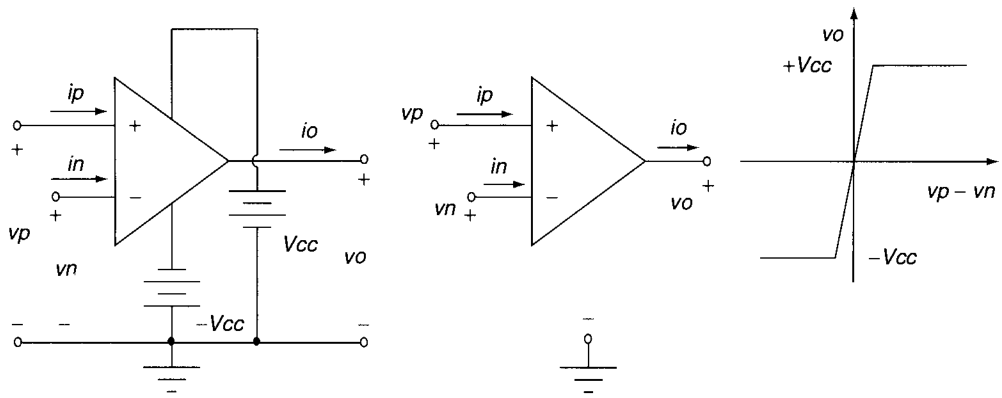

FIGURE 5.3 Multiterminal Nonlinear Circuit Element: An Op-Amp with Five Terminals, its Simplified Circuit Symbol, and an Ideal Characteristic

5.3 Voltages and Currents in Nonlinear Circuits

Circuit voltages and **currents** are solutions to nonlinear differential-algebraic equations that describe circuit behavior. Of particular interest are circuit equilibrium points, which are solutions to associated nonlinear algebraic equations. Hence, circuit equilibrium points are related to dc operating points of a resistive circuit with all capacitors and inductors removed. If capacitor voltages and inductor currents are chosen as state variables, there is a one-to-one correspondence between the circuit's equilibrium points and dc operating points (Chva *et al.*, 1987).

Analyzing nonlinear circuits is often difficult. Only a few simple circuits are adequately described by equations that have a closed form solution. Contrary to linear circuits, which consist of linear elements only (excluding the independent current and voltage sources), nonlinear circuits may possess multiple solutions or may not possess a solution at all (Willson, 1994). A trivial example is a circuit consisting of a current source and an exponential diode, where the value of the dc current supplied by the current source is more negative than the asymptotic value of the current permitted by the diode characteristic when the diode is reverse-biased. It is also possible to construct circuits employing the Ebers-Moll modeled (Ebers and Moll, 1954) bipolar junction transistors whose dc equations may have no solution (Willson, 1995).

5.3.1 Graphical Method for Analysis of Simple Nonlinear Circuits

Voltages and currents in circuits containing only a few nonlinear circuit elements may be found using graphical methods for solving nonlinear equations that describe the behavior of the circuit. A simple nonlinear circuit consisting

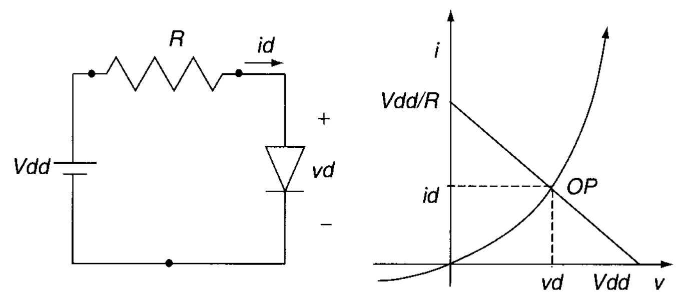

FIGURE 5.4 Simple Nonlinear Circuit and a Graphical Approach for Finding its DC Operating Point (OP): The circuit's load line is obtained by applying Kirchhoff's voltage law. The intersection of diode's exponential characteristic and the load line provides the circuit's dc operating point.

of a constant voltage source, a linear resistor, and an exponential diode is shown in Figure 5.4. Circuit equations can be solved using a graphical method. The solution is the circuit's dc operating point, found as the intersection of the diode characteristics and the "load line." The load line is obtained by applying Kirchhoff's voltage law to the single circuit's loop.

Another simple nonlinear circuit, shown in Figure 5.5, is used to rectify a sinusoidal signal. If the diode is ideal, the sinusoidal signal propagates unaltered and the current flowing through the diode is an ideally rectified sinusoidal signal. The circuit's steady-state response can be found graphically. The diode is *off* below the value $v_s(t) = V_{dd}$. For values $v_s(t) \geq V_{dd}$ the diode is *on*. Hence,

$$\begin{aligned} i_d(t) &= 0 \text{ for } v_s(t) \leq V_{dd}. \\ i_d(t) &= \frac{v_s(t) - V_{dd}}{R} \text{ for } v_s(t) > V_{dd}. \end{aligned} \tag{5.11}$$

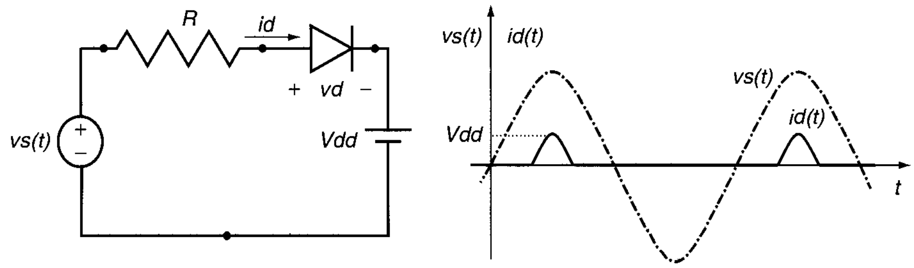

FIGURE 5.5 A Simple Rectifier Circuit with an Ideal Diode: The steady-state solution can be found graphically.

5.3.2 Computer-Aided Tools for Analysis of Nonlinear Circuits

In most cases, a computer program is used to numerically solve nonlinear differential-algebraic circuit equations. The most popular circuits analysis program is SPICE (Massobrio and Antognetti, 1993; Quarles *et al.*, 1994; Vladimirescu, 1994). The original **SPICE code** has been modified and enhanced in numerous electronic design automation tools that employ computer programs to analyze complex integrated circuits containing thousands of nonlinear elements, such as bipolar junction and field-effect transistors.

One of the most important problems when designing a transistor circuit is to find its dc operating point(s) (i.e., voltages and currents in a circuit when all sources are dc sources). The mathematical problem of finding a nonlinear circuit's dc operating points is described by a set of nonlinear algebraic equations constructed by applying Kirchhoff's voltage and current laws and by employing the characteristic of the circuit elements. A common numerical approach for finding these operating points is the Newton–Raphson method and its variants. These methods require a good starting point and sometimes fail. This is known as the dc convergence problem. In the past decade, several numerical techniques based on continuation, parameter-embedding, or homotopy methods have been proposed to successfully solve convergence problems that often appear in circuits possessing more than one dc operating point (Melville *et al.*, 1993; Trajković, 1999; Wolf and Sanders, 1996; Yamamura and Horiuchi, 1990; Yamamura *et al.*, 1999).

5.3.3 Qualitative Properties of Circuit Solutions

Many fundamental issues arise in the case of nonlinear circuits concerning a solution's existence, uniqueness, continuity, boundedness, and stability. Mathematical tools used to address these issues and examine properties of nonlinear circuits range from purely numerical to geometric (Smale, 1972). Briefly addressed here are solutions to a circuit's equations. Described are several fundamental results concerning their properties.

Existence and Uniqueness of Solutions

Circuits with nonlinear elements may have multiple discrete dc operating points (equilibriums). In contrast, circuits consisting of positive linear resistors possess either one dc operating point or, in special cases, a continuous family of dc operating points. Many resistive circuits consisting of independent voltage sources and voltage-controlled resistors, whose v–i relation characteristics are continuous strictly monotone-increasing functions, have at most one solution (Duffin, 1947; Minty, 1960; Willson, 1975). Many transistor circuits possess the same property based on their topology alone. Other circuits, such as flip-flops and memory cells, possess feedback structures (Nielsen and Willson, 1980). These circuits may possess multiple operating points with an appropriate choice of circuit parameters and biasing of transistors (Trajković and Willson, 1992). For example, a circuit containing two bipolar transistors possesses at most three isolated dc operating points (Lee and Willson, 1983). Estimating the number of dc operating points or even their upper bounds for an arbitrary nonlinear circuit is still an open problem (Lagarias and Trajković, 1999).

Continuity and Boundedness of Solutions

Properties such as continuity and boundedness of solutions are often desired in cases of well-behaved circuits. One would like to expect that "small" changes in the circuit's output will result from the "small" change of the circuit's input and that the bounded circuit's input leads to the bounded circuit's output. They are intimately related to the stability of circuit solutions.

Stability of Solutions

Once a solution to a circuit's currents and voltages is found, a fundamental question arises regarding its stability. A solution may be stable or unstable, and stability is both a local and a global concept (Hasler and Neirynck, 1986). Local stability usually refers to the solution that, with sufficiently close initial conditions, remains close to other solutions as time increases. A stronger concept of stability is asymptotic stability: if initial conditions are close, the solutions converge toward each other as time approaches infinity. Although no rate of convergence is implied in the case of asymptotic stability, a stronger concept of exponential stability implies that the rate of convergence toward a stable solution is bounded from above by an exponential function. Note that the exponential convergence to a solution does not necessarily imply its stability. Contrary to nonlinear circuits, stability (if it exists) of linear time-invariant circuits is always exponential. Linear time-invariant circuits whose natural frequencies have negative real parts have stable solutions. A global asymptotic (complete) stability implies that no matter what the initial conditions are, the solutions converge toward each other. Hence, a completely stable circuit possesses exactly one solution.

A fundamental observation can be made that there exist dc operating points of transistor circuits that are unstable: if the circuit is biased at such an operating point and if the circuit is augmented with *any* configuration of positive-valued shunt capacitors and/or series inductors, the equilibrium point of the resulting dynamic circuit will always be unstable (Green and Willson, 1992, 1994). A simple example is the common latch circuit that, for appropriate parameter values, possesses three dc operating points: two stable and one unstable.

5.4 Open Problems

In the area of analysis of resistive circuits, interesting and still unresolved issues are: Does a given transistor circuit possess exactly one, or more than one dc operating point? How many operating points can it possess? What simple techniques can be used to distinguish between those circuits having a unique operating point and those capable of possessing more than one? What can we say about the stability of an operating point? How can circuit simulators be used to find all the solutions of a given circuit? These and other issues have been and still are a focal point of research in the area of nonlinear circuits. The references cited provide more detailed discussions of the many topics considered in this elementary review of nonlinear circuits.

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