I

CIRCUIT THEORY

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Circuit theory is an important and perhaps the oldest branch of electrical engineering. A circuit is an interconnection of electrical elements. These include passive elements, such as resistances, capacitances, and inductances, as well as active elements and sources (or excitations). Two variables, namely voltage and current variables, are associated with each circuit element. There are two aspects to circuit theory: **analysis and design**. Circuit analysis involves the determination of current and voltage values in different elements of the circuit, given the values of the sources or excitations. On the other hand, circuit design focuses on the design of circuits that exhibit a certain prespecified voltage or current characteristics at one or more parts of the circuit. Circuits can also be broadly classified as **linear or nonlinear circuits**.

This section consists of five chapters that provide a broad introduction to most fundamental principles and techniques in circuit analysis and design:

- Linear Circuit Analysis
- Circuit Analysis: A Graph-Theoretic Foundation
- Computer-Aided Design
- Synthesis of Networks
- Nonlinear Circuits.

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Linear Circuit Analysis

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1.1 Definitions and Terminology

An **electric charge** is a physical property of electrons and protons in the atoms of matter that gives rise to forces between atoms. The charge is measured in coulomb [C]. The charge of a proton is arbitrarily chosen as positive and has the value of 1.601×10^{-19} C, whereas the charge of an electron is chosen as negative with a value of -1.601×10^{-19} C. Like charges repel while unlike charges attract each other. The electric charges obey the principle of conservation (i.e., charges cannot be created or destroyed).

A **current** is the flow of electric charge that is measured by its flow rate as coulombs per second with the units of ampere [A]. An ampere is defined as the flow of charge at the rate of one coulomb per second (1 A = 1 C/s). In other words, current $i(t)$ through a cross section at time t is given by dq/dt, where

$q(t)$ is the charge that has flown through the cross section up to time t:

$$i(t) = \frac{dq(t)}{dt} [\text{A}]. \tag{1.1}$$

Knowing i, the total charge, Q, transferred during the time from t_1 to t_2 can be calculated as:

$$Q = \int_{t1}^{t2} idt \ [\text{C}]. \tag{1.2}$$

The voltage or potential difference (V_{AB}) between two points A and B is the amount of energy required to move a unit positive charge from B to A. If this energy is positive, that is work is done by external sources against forces on the charges, then V_{AB} is positive and point A is at a higher potential with respect to B. The voltage is measured using the unit of volt [V]. The voltage between two points is 1 V if 1 J (joule) of work is required to move 1 C of charge. If the voltage, v, between two points is constant, then the work, w, done in moving q coulombs of charge between the two points is given by:

$$w = vq \ [\text{J}]. \tag{1.3}$$

Power (p) is the rate of doing work or the energy flow rate. When a charge of dq coulombs is moved from point A to point B with a potential difference of v volts, the energy supplied to the charge will be $v\ dq$ joule [J]. If this movement takes place in dt seconds, the power supplied to the charge will be $v\ dq/dt$ watts [W]. Because dq/dt is the charge flow rate defined earlier as current i, the power supplied to the charge can be written as:

$$p = vi \ [\text{W}]. \tag{1.4}$$

The energy supplied over duration $t1$ to $t2$ is then given by:

$$w = \int_{t1}^{t2} vi\,dt \ [\text{J}]. \tag{1.5}$$

A **lumped electrical element** is a model of an electrical device with two or more terminals through which current can flow in or out; the flow can pass *only* through the terminals. In a two-terminal element, current flows through the element entering via one terminal and leaving via another terminal. On the other hand, the voltage is present across the element and measured between the two terminals. In a multiterminal element, current flows through one set of terminals and leaves through the remaining set of terminals. The relation between the voltage and current in an element, known as the v–i relation, defines the element's characteristic. A circuit is made up of electrical elements.

Linear elements include a v–i relation, which can be linear if it satisfies the homogeneity property and the superposition principle. The homogeneity property refers to proportionality; that is, if i gives a voltage of v, ki gives a voltage of kv for any arbitrary constant k. The superposition principle implies additivity; that is, if i_1 gives a voltage of v_1 and i_2 gives a voltage of v_2, then $i_1 + i_2$ should give a voltage $v_1 + v_2$. It is easily verified that $v = Ri$ and $v = L\ di/dt$ are linear relations. Elements that possess such linear relations are called linear elements, and a circuit that is made up of linear elements is called a linear circuit.

Sources, also known as **active elements,** are electrical elements that provide power to a circuit. There are two types of sources: (1) independent sources and (2) dependent (or controlled) sources. An independent voltage source provides a specified voltage irrespective of the elements connected to it. In a similar manner, an independent current source provides a specified current irrespective of the elements connected to it. Figure 1.1 shows representations of independent voltage and independent current sources. It may be noted that the value of an independent voltage or an independent current source may be constant in magnitude and direction (called a direct current [dc] source) or may vary as a function of time (called a time-varying source). If the variation is of sinusoidal nature, it is called an alternating current (ac) source.

Values of dependent sources depend on the voltage or current of some other element or elements in the circuit. There are four classes of dependent sources: (1) voltage-controlled voltage source, (2) current-controlled voltage source, (3) voltage-controlled current source and (4) current-controlled current source. The representations of these dependent sources are shown in Table 1.1.

Passive elements consume power. Names, symbols, and the characteristics of some commonly used passive elements are given in Table 1.2. The v–i relation of a linear resistor, $v = Ri$,

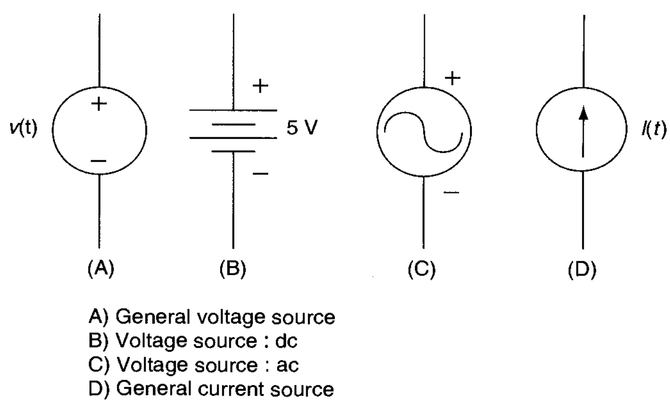


FIGURE 1.1 Independent Voltage and Current Sources

TABLE 1.1 Dependent Sources and Their Representation

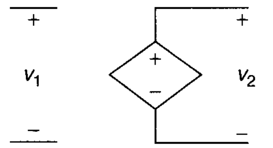
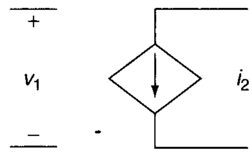
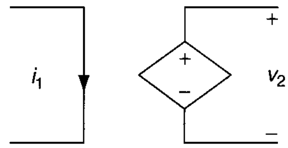
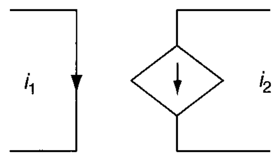
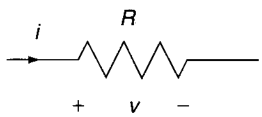
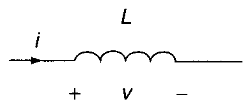
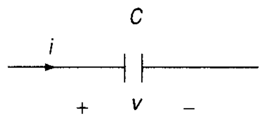
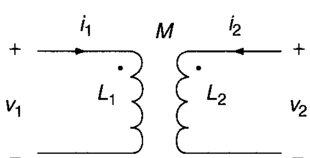
Element	Voltage and current relation	Representation
Voltage-controlled voltage source	$v_2 = a\, v_1$ a: Voltage gain	
Voltage-controlled current source	$i_2 = g_t v_1$ g_t: Transfer conductance	
Current-controlled voltage source	$v_2 = r_t i_1$ rt: Transfer resistance	
Current-controlled current source	$i_2 = b i_1$ b: Current gain	

TABLE 1.2 Some Passive Elements and Their Characteristics

Name of the element	Symbol	The v–i relation	Unit
Resistance: R		$v = Ri$	ohm [Ω]
Inductance: L		$v = L\, di/dt$	henry [H]
Capacitance: C		$i = C\, dv/dt$	farad [F]
Mutual Inductance: M		$v_1 = M\, di_2/dt + L_1\, di_1/dt$ $v_2 = M\, di_1/dt + L_2\, di_2/dt$	henry [H]

is known as Ohm's law, and the linear relations of other passive elements are sometimes called generalized Ohm's laws. It may be noted that in a passive element, the polarity of the voltage is such that current flows from positive to negative terminals. This polarity marking is said to follow the passive polarity convention.

A **circuit** is formed by an interconnection of circuit elements at their terminals. A **node** is a junction point where the

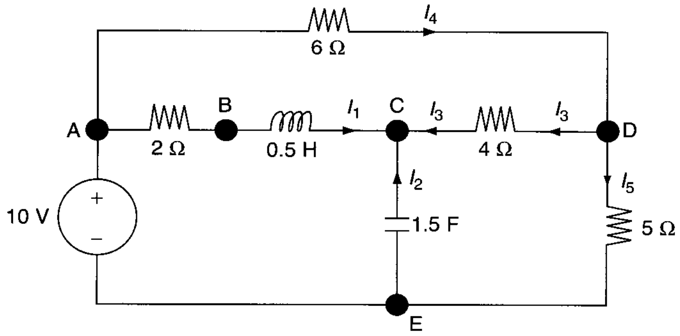


FIGURE 1.2 Example Circuit Diagram

terminals of two or more elements are joined. Figure 1.2 shows A, B, C, D, and E as nodes. A **loop** is a closed path in a circuit such that each node is traversed only once when tracing the loop. In Figure 1.2, ABCEA is a loop, and ABCDEA is also a loop. A **mesh** is a special class of loop that is associated with a window of a circuit drawn in a plane (**planar circuit**). In the same Figure ABCEA is a mesh, whereas ABCDEA is not considered a mesh for the circuit as drawn. A **network** is defined as a circuit that has a set of terminals available for external connections (i.e., accessible from outside of the circuit). A **pair of terminals** of a network to which a source, another network, or a measuring device can be connected is called a **port** of the network. A network containing such a pair of terminals is called a **one-port network.** A network containing two pairs of externally accessible terminals is called a **two-port network,** and multiple pairs of externally accessible terminal pairs are called a **multiport network.**

1.2 Circuit Laws

Two important laws are based on the physical properties of electric charges, and these laws form the foundation of circuit analysis. They are Kirchhoff's current law (KVL) and Kirchhoff's voltage law (KCL). While Kirchhoff's current law is based on the principle of conservation of electric charge, Kirchhoff's voltage law is based on the principle of energy conservation.

1.2.1 Kirchhoff's Current Law

At any instant, the algebraic sum of the currents (i) entering a node in a circuit is equal to zero. In the circuit in Figure 1.2, application of KCL at node C yields the following equation:

$$i_1 + i_2 + i_3 = 0 \tag{1.6}$$

Similarly at node D, KCL yields:

$$i_4 - i_3 - i_5 = 0. \tag{1.7}$$

1.2.2 Kirchhoff's Voltage Law

At any instant, the algebraic sum of the voltages (v) around a loop is equal to zero. In going around a loop, a useful convention is to take the voltage drop (going from positive to negative) as positive and the voltage rise (going from negative to positive) as negative. In Figure 1.2, application of KVL around the loop ABCEA gives the following equation:

$$v_{AB} + v_{BC} + v_{CE} + v_{EA} = 0. \tag{1.8}$$

1.3 Circuit Analysis

Analysis of an electrical circuit involves the determination of voltages and currents in various elements, given the element values and their interconnections. In a linear circuit, the v–i relations of the circuit elements and the equations generated by the application of KCL at the nodes and of KVL for the loops generate a sufficient number of simultaneous linear equations that can be solved for unknown voltages and currents. Various steps involved in the analysis of linear circuits are as follows:

1. For all the elements except the current sources, assign a current variable with arbitrary polarity. For the current sources, current values and polarity are given.
2. For all elements except the voltage sources, assign a voltage variable with polarities based on the passive sign convention. For voltage sources, the voltages and their polarities are known.
3. Write KCL equations at $N - 1$ nodes, where N is the total number of nodes in the circuit.
4. Write expressions for voltage variables of passive elements using their v–i relations.
5. Apply KVL equations for $E - N + 1$ independent loops, where E is the number of elements in the circuit. In the case of planar circuits, which can be drawn on a plane paper without edges crossing over one another, the meshes will form a set of independent loops. For nonplanar circuits, use special methods that employ topological techniques to find independent loops.
6. Solve the $2E$ equations to find the E currents and E voltages.

The following example illustrates the application of the steps in this analysis.

Example 1.1. *For the circuit in Figure 1.3, determine the voltages across the various elements.* Following step 1, assign the currents I_1, I_2, I_3, and I_4 to the elements. Then apply the KCL to the nodes A, B, and C to get $I_4 - I_1 = 0$, $I_1 - I_2 = 0$, and $I_2 - I_3 = 0$. Solving these equations produces $I_1 = I_2 = I_3 = I_4$. Applying the v–i relation characteristics of the nonsource elements, you get $V_{AB} = 2\, I_1$, $V_{BC} = 3\, I_2$, and $V_{CD} = 5\, I_3$. Applying

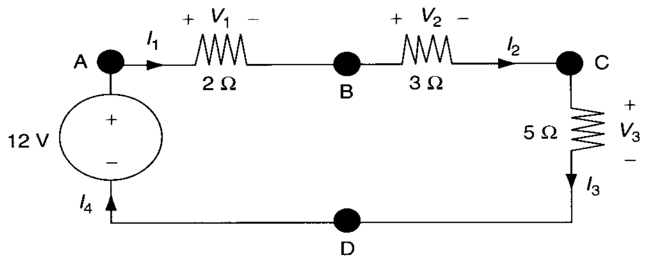


FIGURE 1.3 Circuit for Example 1.1

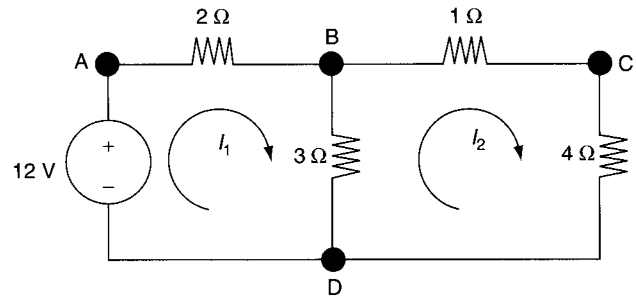


FIGURE 1.4 Circuit for Example 1.2

the KVL to the loop ABCDA, you determine $V_{AB} + V_{BC} + V_{CD} + V_{DA} = 0$. Substituting for the voltages in terms of currents, you get $2\ I_1 + 3\ I_1 + 5\ I_1 - 12 = 0$. Simplifying results in $10\ I_1 = 12$ to make $I_1 = 1.2$ A. The end results are $V_{AB} = 2.4\,\text{V}$, $V_{BC} = 3.6\,\text{V}$, and $V_{CD} = 6.0\,\text{V}$.

In the above circuit analysis method, 2E equations are first set up and then solved simultaneously. For large circuits, this process can become very cumbersome. Techniques exist to reduce the number of unknowns that would be solved simultaneously. Two most commonly used methods are the loop current method and the node voltage method.

1.3.1 Loop Current Method

In this method, one distinct current variable is assigned to each independent loop. The element currents are then calculated in terms of the loop currents. Using the element currents and values, element voltages are calculated. After these calculations, Kirchhoff's voltage law is applied to each of the loops, and the resulting equations are solved for the loop currents. Using the loop currents, element currents and voltages are then determined. Thus, in this method, the number of simultaneous equations to be solved are equal to the number of independent loops. As noted above, it can be shown that this is equal to $E - N + 1$. Example 1.2 illustrates the techniques just discussed. It may be noted that in the case of planar circuits, the meshes can be chosen as the independent loops.

EXAMPLE 1.2. *In the circuit in Figure 1.4, find the voltage across the 3-Ω resistor.* First, note that there are two independent loops, which are the two meshes in the circuit, and that loop currents I_1 and I_2 are assigned as shown in the diagram. Then calculate the element currents as $I_{AB} = I_1$, $I_{BC} = I_2$, $I_{CD} = I_2$, $I_{BD} = I_1 - I_2$, and $I_{DA} = I_1$. Calculate the element voltages as $V_{AB} = 2\ I_{AB} = 2\ I_1$, $V_{BC} = 1\ I_{BC} = 1\ I_2$, $V_{CD} = 4\ I_2$, and $V_{BD} = 3\ I_{BD} = 3(I_1 - I_2)$. Applying KVL to loops 1 (ABDA) and 2 (BCDB) and substituting the voltages in terms of loop currents results in:

$$5\ I_1 - 3\ I_2 = 12$$
$$-3\ I_1 + 8\ I_2 = 0.$$

Solving the two equations, you get $I_1 = 96/31$ A and $I_2 = 36/31$ A. The voltage across the 3-Ω resistor is $3(I_1 - I_2) = 3(96/31 - 36/31) = 180/31$ A.

Special case 1

When one of the elements in a loop is a current source, the voltage across it cannot be written using the *v–i* relation of the element. In this case, the voltage across the current source should be treated as an unknown variable to be determined. If a current source is present in only one loop and is not common to more than one loop, then the current of the loop in which the current source is present should be equal to the value of the current source and hence is known. To determine the remaining currents, there is no need to write the KVL equation for the current source loop. However, to determine the voltage of the current source, a KVL equation for the current source loop needs to be written. This equation is presented in example 1.3.

EXAMPLE 1.3. *Analyze the circuit shown in Figure 1.5 to find the voltage across the current sources.* The loop currents are assigned as shown. It is easily seen that $I_3 = -2$. Writing KVL equations for loops 1 and 2, you get:

Loop 1: $2(I_1 - I_2) + 4(I_1 - I_3) - 14 = 0 \Rightarrow$
$6\ I_1 - 2\ I_2 = 6.$
Loop 2: $I_2 + 3(I_2 - I_3) + 2(I_2 - I_1) = 0 \Rightarrow$
$-2\ I_1 + 6\ I_2 = -6.$

Solving the two equations simultaneously, you get $I_1 = 3/4$ A and $I_2 = -3/4$ A. To find the V_{CD} across the current source, write the KVL equation for the loop 3 as:

$$4(I_3 - I_1) + 3(I_3 - I_2) + V_{CD} = 0 \Rightarrow$$
$$V_{CD} = 4\ I_1 + 3\ I_2 - 7\ I_3 = 14.75\ \text{V}.$$

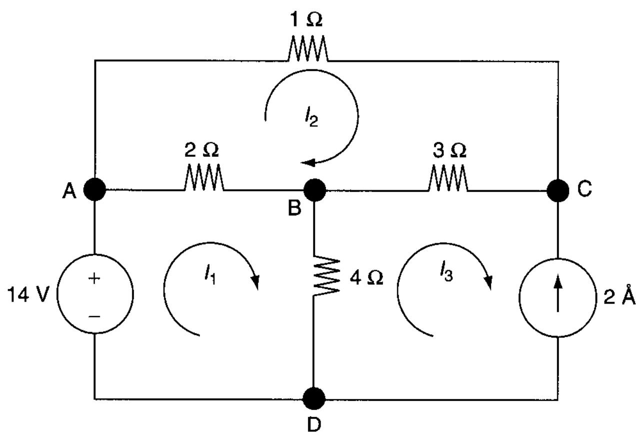


FIGURE 1.5 Circuit for Example 1.3

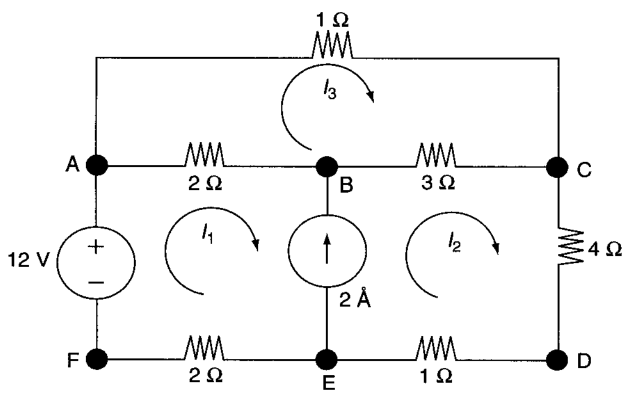


FIGURE 1.6 Circuit for Example 1.4

Special case 2

This case concerns a current source that is common to more than one loop. The solution to this case is illustrated in example 1.4.

EXAMPLE 1.4. *In the circuit shown in Figure 1.6, the 2 A current source is common to loops 1 and 2.* One method of writing KVL equations is to treat V_{BE} as an unknown and write three KVL equations. In addition, you can write the current of the current source as $I_2 - I_1 = 2$, giving a fourth equation. Solving the four equations simultaneously, you determine the values of I_1, I_2, I_3, and V_{BE}. These equations are the following:

Loop 1: $2(I_1 - I_3) + V_{BE} + 2\ I_1 - 12 = 0$
$=> 4\ I_1 - 2\ I_3 + V_{BE} = 12.$
Loop 2: $3(I_2 - I_3) + 4\ I_2 + I_2 - V_{BE} = 0$
$=> 8\ I_2 - 3\ I_3 - V_{BE} = 0.$
Loop 3: $I_3 + 3(I_3 - I_2) + 2(I_3 - I_1) = 0$
$=> -2\ I_1 - 3\ I_2 + 6\ I_3 = 0.$
Current source relation: $-I_1 + I_2 = 2.$

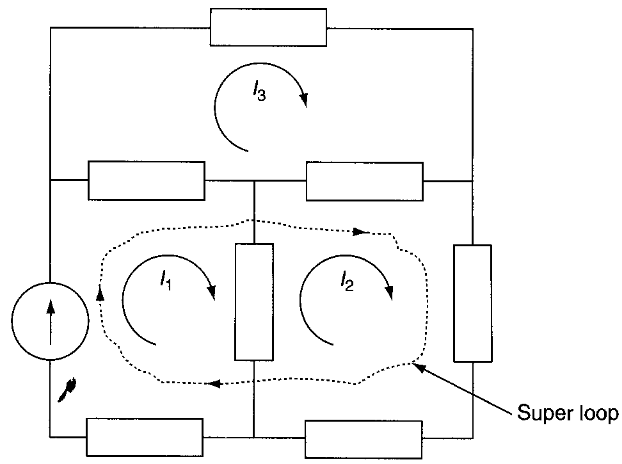


FIGURE 1.7 Circuit in Figure 1.6 with the Super Loop Shown as Dotted Line

Solving the above four equations results in $I_1 = 0.13$ A, $I_2 = 2.13$ A, $I_3 = 1.11$ A, and $V_{BE} = 13.70$ V.

Alternative method for special case 2 (Super loop method): This method eliminates the need to add the voltage variable as an unknown. When a current source is common to loops 1 and 2, then KVL is applied on a new loop called the **super loop**. The super loop is obtained from combining loops 1 and 2 (after deleting the common elements) as shown in Figure 1.7. For the circuit considered in example 1.4, the loop ABCDEFA is the super loop obtained by combining loops 1 and 2. The KVL is applied on this super loop instead of KVL being applied for loop 1 and loop 2 separately. The following is the KVL equation for super loop ABCDEFA:

$$2(I_1 - I_3) + 3(I_2 - I_3) + 4\ I_2 + I_2 + 2\ I_1 - 12 = 0$$
$$=> 4\ I_1 + 8\ I_2 - 5\ I_3 = 12.$$

The KVL equation around loop 3 is written as:

$$-2\ I_1 - 3\ I_2 + 6\ I_3 = 0.$$

The current source can be written as:

$$-I_1 + I_2 = 2.$$

Solving the above three equations simultaneously produces equations $I_1 = 0.13$ A, $I_2 = 2.13$ A, and $I_3 = 1.11$ A.

1.3.2 Node Voltage Method (Nodal Analysis)

In this method, one node is chosen as the reference node whose voltage is assumed as zero, and the voltages of other nodes are expressed with respect to the reference node. For example, in Figure 1.8, the voltage of node G is chosen as the

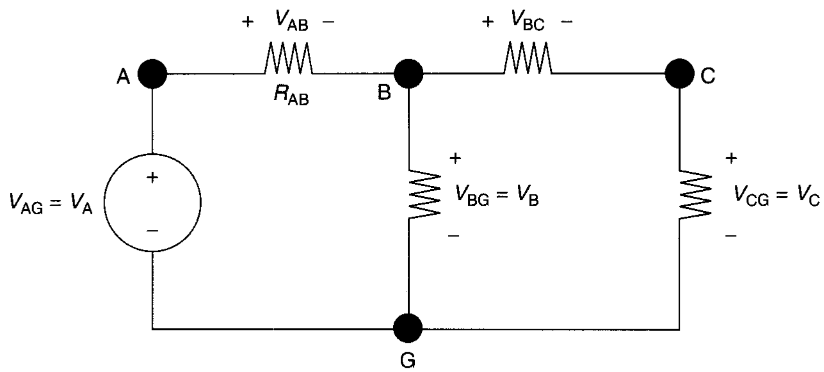


FIGURE 1.8 Circuit with Node Voltages Marked

reference node, and then the voltage of node A is $V_A = V_{AG}$ and that of node B is $V_B = V_{BG}$ and so on. Then, for every element between two nodes, the element voltages may be expressed as the difference between the two node voltages. For example, the voltage of element R_{AB} is $V_{AB} = V_A - V_B$. Similarly $V_{BC} = V_B - V_C$ and so on. Then the current through the element R_{AB} can be determined using the v–i characteristic of the element as $I_{AB} = V_{AB}/R_{AB}$. Once the currents of all elements are known in terms of node voltages, KCL is applied for each node except for the reference node, obtaining a total of N–1 equations where N is the total number of nodes.

Special Case 1

In branches where voltage sources are present, the v–i relation cannot be used to find the current. Instead, the current is left as an unknown. Because the voltage of the element is known, another equation can be used to solve the added unknown. When the element is a current source, the current through the element is known. There is no need to use the v–i relation. The calculation is illustrated in the following example.

EXAMPLE 1.5. In Figure 1.9, solve for the voltages V_A, V_B, and V_C with respect to the reference node G. At node A, $V_A = 12$. At node B, KCL yields:

$$I_{BA} + I_{BG} + I_{BC} = 0 \Rightarrow$$
$$(V_B - V_A)/1 + V_B/4 + (V_B - V_C)/5 = 0 \Rightarrow$$
$$-V_A + (1 + 1/4 + 1/5)V_B - V_C/5 = 0.$$

Similarly at node C, KCL yields:

$$V_A/2 - V_B/5 + (1/5 + 1/2)V_C = 2.$$

Solving the above three equations simultaneously results in $V_A = 12$ V, $V_B = 10.26$ V, and $V_C = 14.36$ V.

Super Node: When a voltage source is present between two nonreference nodes, a super node may be used to avoid introducing an unknown variable for the current through the voltage source. Instead of applying KCL to each of the two nodes of the voltage source element, KCL is applied to an imaginary node consisting of both the nodes together. This imaginary node is called a super node. In Figure 1.10, the super node is shown by a dotted closed shape. KCL on this super node is given by:

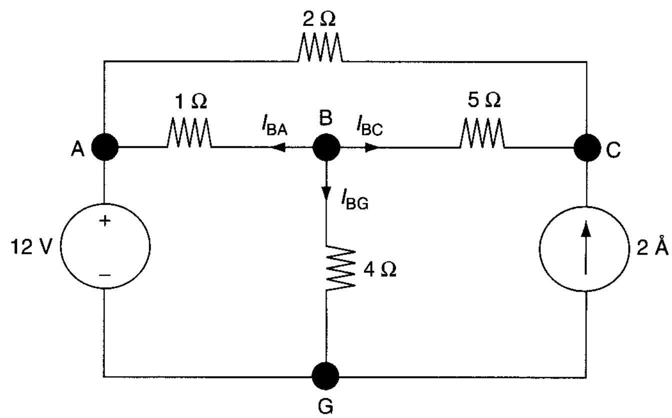


FIGURE 1.9 Circuit for Example 1.5

$$I_{BA} + I_{BG} + I_{CG} + I_{CA} = 0 \Rightarrow (V_B - V_A)/1$$
$$+ V_B/3 + V_C/4 + (V_C - V_A)/2 = 0.$$

In addition to this equation, the two voltage constraint equations, $V_A = 10$ and $V_B - V_C = 5$, are used to solve for V_B and V_C as $V_B = 9$ V and $V_C = 4$ V.

1.4 Equivalent Circuits

Two linear circuits, say circuit 1 and circuit 2, are said to be equivalent across a specified set of terminals if the voltage–current relations for the two circuits across the specified terminals are identical. Now consider a composite circuit

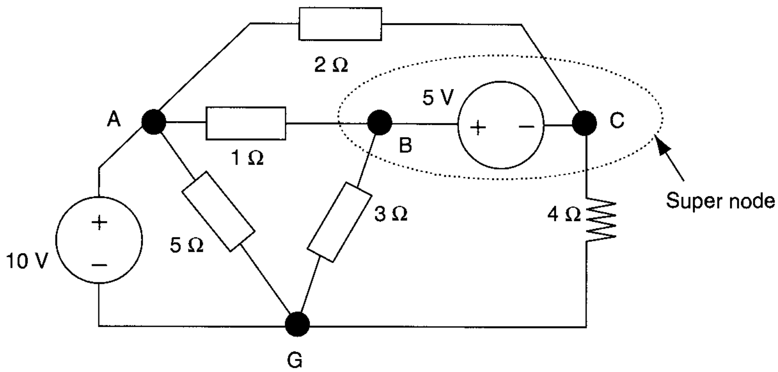


FIGURE 1.10 Circuit with Super Node

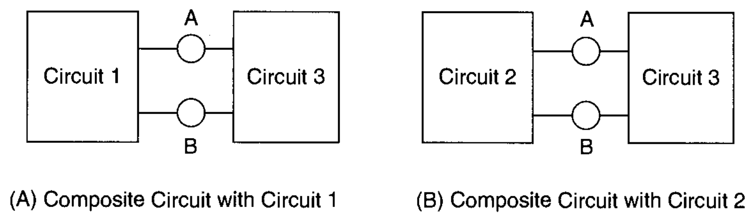


FIGURE 1.11 Equivalent Circuit Application

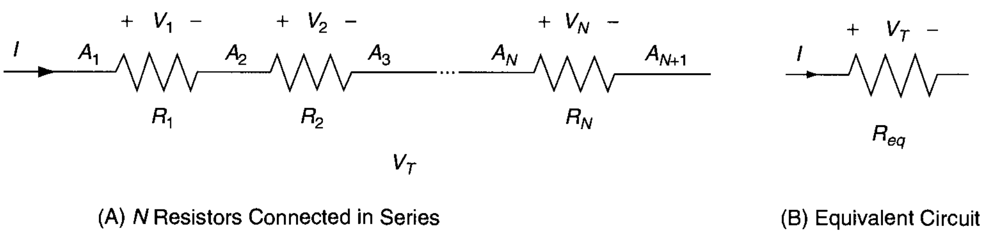


FIGURE 1.12 Resistances Connected in Series

consisting of circuit 1 connected to another circuit, circuit 3, at the specified terminals as shown in Figure 1.11(A). The voltages and currents in circuit 3 are not altered if circuit 2 replaces circuit 1, as shown in Figure 1.11(B). If circuit 2 is simpler than circuit 1, then the analysis of the composite circuit will be simplified. A number of techniques for obtaining two-terminal equivalent circuits are outlined in the following section.

1.4.1 Series Connection

Two two-terminal elements are said to be **connected in series** if the connection is such that the same current flows through both the elements as shown in Figure 1.12. When two resistances R_1 and R_2 are connected in series, they can be replaced by a single element having an equivalent resistance of sum of the two resistances, $R_{eq} = R_1 + R_2$, without affecting the voltages and currents in the rest of the circuit. In a similar manner, if N resistances $R_1, R_2, \ldots, R_N$ are connected in series, their equivalent resistance will be given by:

$$R_{eq} = R_1 + R_2 + \ldots + R_N. \tag{1.9}$$

Voltage Division: When a voltage V_T is present across N resistors connected in series, the total voltage divides across the resistors proportional to their resistance values. Thus

$$V_1 = V_T \frac{R_1}{R_{eq}}, \; V_2 = V_T \frac{R_2}{R_{eq}}, \ldots, \; V_N = V_T \frac{R_N}{R_{eq}}, \tag{1.10}$$

where $R_{eq} = R_1 + R_2 + \ldots + R_N$.

1.4.2 Parallel Connection

Two-terminal elements are said to be **connected in parallel** if the same voltage exists across all the elements and if they have two distinct common nodes as shown in Figure 1.13. In the case of a parallel connection, conductances, which are reciprocals of resistances, sum to give an equivalent conductance of G_{eq}:

$$G_{eq} = G_1 + G_2 + \ldots + G_N, \tag{1.11}$$

or equivalently

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_N}. \tag{1.12}$$

Current Division: In parallel connection, the total current I_T of the parallel combination divides proportionally to the conductance of each element. That is, the current in each element is proportional to its conductance and is given by:

$$I_1 = I_T\frac{G_1}{G_{eq}},\ I_2 = I_T\frac{G_2}{G_{eq}},\ \ldots,\ I_N = I_T\frac{G_N}{G_{eq}}, \tag{1.13}$$

where $G_{eq} = G_1 + G_2 + \ldots + G_N$.

1.4.3 Star–Delta (Wye–Delta or T–Pi) Transformation

It can be shown that the star subnetwork connected as shown in Figure 1.14 can be converted into an equivalent delta subnetwork. The element values between the two subnetworks are related as shown in Table 1.3. It should be noted that the star subnetwork has four nodes, whereas the delta network has only three nodes. Hence, the star network can be replaced in a circuit without affecting the voltages and currents in the rest

TABLE 1.3 Relations Between the Element Values in Star and Delta Equivalent Circuits

Star in terms of delta resistances	Delta in terms of star resistances
$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$	$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$
$R_2 = \frac{R_c R_a}{R_a + R_b + R_c}$	$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$
$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$	$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$

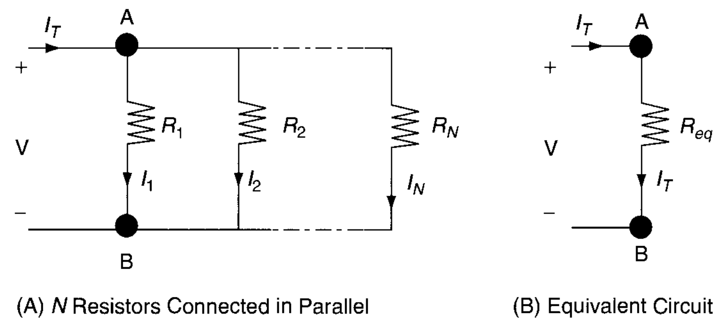


FIGURE 1.13 Resistances Connected in Parallel

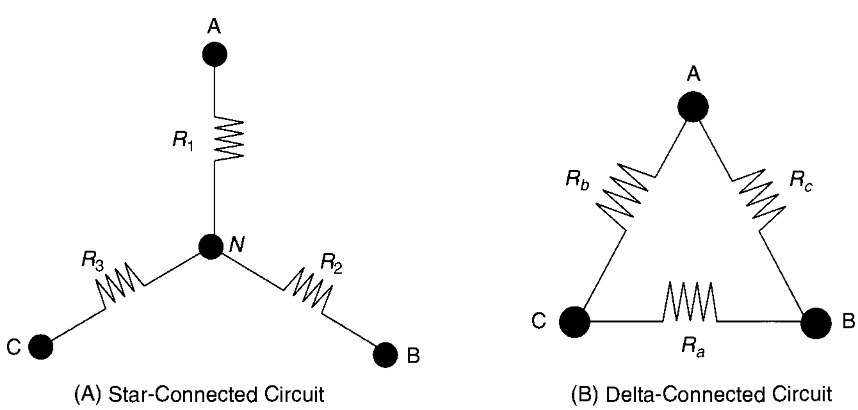


FIGURE 1.14 Star and Delta Equivalent Circuits

of the circuit *only* if the central node in the star subnetwork is not connected to any other circuit node.

1.4.4 Thevenin Equivalent Circuit

A network consisting of linear resistors and dependent and independent sources with a pair of accessible terminals can be represented by an equivalent circuit with a voltage source and a series resistance as shown in Figure 1.15. V_{TH} is equal to the open circuit voltage across the two terminals A and B, and R_{TH} is the resistance measured across nodes A and B (also called **looking-in resistance**) when the independent sources in the network are deactivated. The R_{TH} can also be determined as $R_{TH} = V_{oc}/I_{sc}$, where V_{oc} is the open circuit voltage across terminals A and B and where I_{sc} is the short circuit current that will flow from A to B through an external zero resistance connection (short circuit) if one is made.

1.4.5 Norton Equivalent Circuit

A two-terminal network consisting of linear resistors and independent and dependent sources can be represented by an equivalent circuit with a current source and a parallel resistor as shown in Figure 1.16. In this figure, I_N is equal to the short circuit current across terminals A and B, and R_N is the looking-in resistance measured across A and B after the independent sources are deactivated. It is easy to see the following relation between Thevenin equivalent circuit parameters and the Norton equivalent circuit parameters:

$$R_N = R_{TH} \text{ and } I_N = V_{TH}/R_{TH}. \tag{1.14}$$

1.4.6 Source Transformation

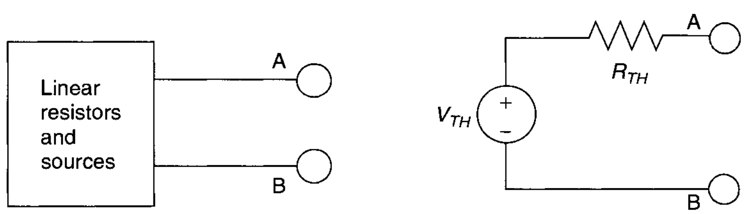
Using a Norton equivalent circuit, a voltage source with a series resistor can be converted into an equivalent current source with a parallel resistor. In a similar manner, using Thevenin theorem, a current source with a parallel resistor can be represented by a voltage source with a series resistor. These transformations are called source transformations. The two sources in Figure 1.17 are equivalent between nodes B and C.

1.5 Network Theorems

A number of theorems that simplify the analysis of linear circuits have been proposed. The following section presents, without proof, two such theorems: the superposition theorem and the maximum power transfer theorem.

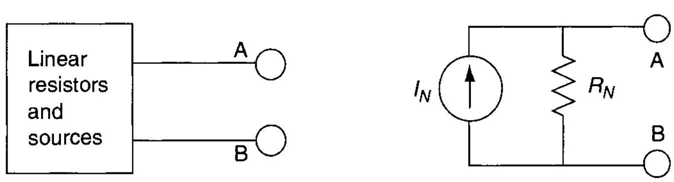
1.5.1 Superposition Theorem

For a circuit consisting of linear elements and sources, the response (voltage or current) in any element in the circuit is the algebraic sum of the responses in this element obtained by applying one independent source at a time. When one independent source is applied, all other independent sources are deactivated. It may be noted that a deactivated voltage source behaves as a short circuit, whereas a deactivated current source behaves as an open circuit. It should also be noted that the dependent sources in the circuit are not deactivated. Further, any initial condition in the circuit is treated as an appropriate independent source. That is, an initially charged



(A) Linear Network with Two Terminals (B) Equivalent Circuit Across the Terminals

FIGURE 1.15 Thevenin Equivalent Circuit



(A) Linear Network with Two Terminals (B) Equivalent Circuit Across AB in (A)

FIGURE 1.16 Norton Equivalent Circuit

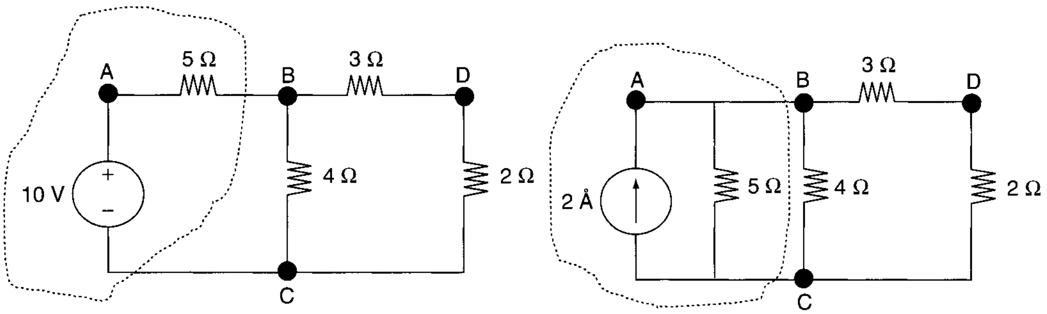


FIGURE 1.17 Example of Source Transformation

capacitor is replaced by an uncharged capacitor in series with an independent voltage source. Similarly, an inductor with an initial current is replaced with an inductor without any initial current in parallel with an independent current source. The following example illustrates the application of superposition in the analysis of linear circuits.

EXAMPLE 1.6. *For the circuit in Figure 1.18(A), determine the voltage across the 3-Ω resistor.* The circuit has two independent sources, one voltage source and one current source. Figure 1.18(B) shows the circuit when voltage source is activated and current source is deactivated (replaced by an open circuit). Let V_{31} be the voltage across the 3-Ω resistor in this circuit. Figure 1.18(C) shows the circuit when current source is activated and voltage source is deactivated (replaced by a short circuit). Let V_{32} be the voltage across the 3-Ω resistor in this circuit. Then you determine that the voltage across the 3-Ω resistor in the given complete circuit is $V_3 = V_{31} + V_{32}$.

1.5.2 Maximum Power Transfer Theorem

In the circuit shown in Fig. 1.19, power supplied to the load is maximum when the load resistance is equal to the source resistance.

It may be noted that the application of the maximum power transfer theorem is not restricted to simple circuits only. The theorem can also be applied to complicated circuits as long as the circuit is linear and there is one variable load. In such cases, the complicated circuit across the variable load is replaced by its equivalent Thevenin circuit. The maximum power transfer theorem is then applied to find the load resistance that leads to maximum power in the load.

1.6 Time Domain Analysis

When a circuit contains energy storing elements, namely inductors and capacitors, the analysis of the circuit involves the solution of a differential equation.

1.6.1 First-Order Circuits

A circuit with a single energy-storing element yields a first-order differential equation as shown below for the circuits in Figure 1.20.

Consider the RC circuit in Figure 1.20(A). For $t > 0$, writing KVL around the loop, the result is equation:

$$Ri + v_c(0) + \frac{1}{c}\int_0^t idt = v_s. \tag{1.15}$$

Differentiating with respect to t yields:

$$R\frac{di}{dt} + \frac{1}{c}i = 0. \tag{1.16}$$

The solution of the above homogeneous differential equation can be obtained as:

$$i(t) = Ke^{-1/RCt}. \tag{1.17}$$

The value of K can be found by using the initial voltage $v_c(0)$ in the capacitor as:

$$K = i(0) = \frac{v_s - v_c(0)}{R}. \tag{1.18}$$

Substituting this value in the expression for $i(t)$ determines the final solution for $i(t)$ as:

$$i(t) = \frac{v_s - v_c(0)}{R}e^{-(1/RC)t}. \tag{1.19}$$

This exponentially decreasing response $i(t)$ is shown in Figure 1.21(A). It has a time constant of $\tau = RC$. The voltage $V_c(t)$ is also shown in Figure 1.21(B)

In a similar manner, the differential equation for $i(t)$ in the RL circuit shown in Figure 1.20(B) can be obtained for $t > 0$ as:

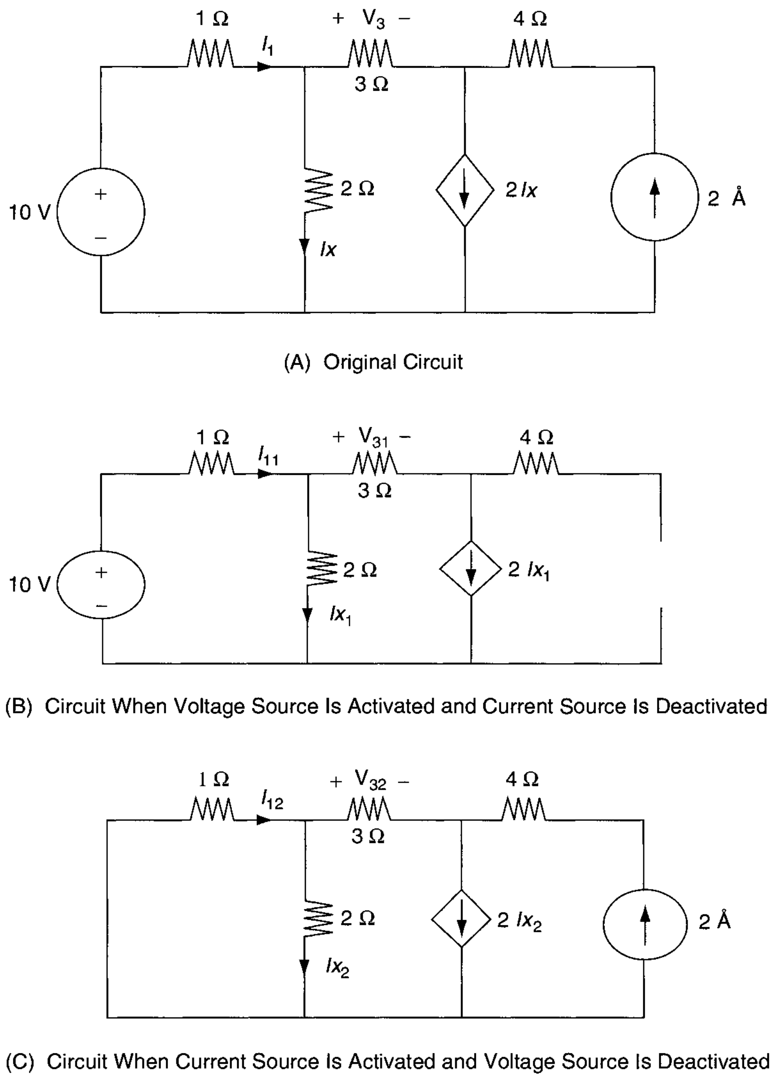


FIGURE 1.18 Circuits for Example 1.6

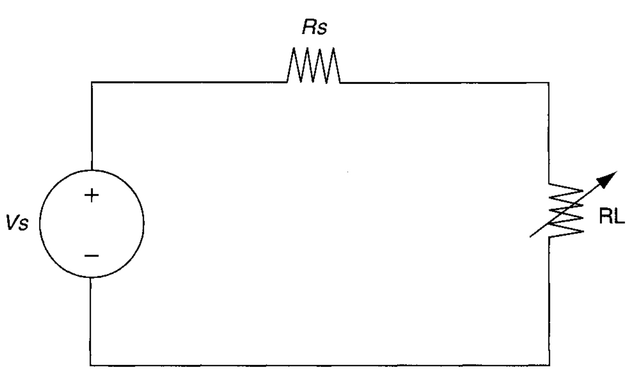


FIGURE 1.19 Circuit with a Variable Load Excited by a Thevenin Source

$$Ri + L\frac{di}{dt} = v_s. \tag{1.20}$$

Because this is a nonhomogeneous differential equation, its solution consists of two parts:

$$i(t) = i_n(t) + i_f(t), \tag{1.21}$$

where $i_n(t)$, the natural response (also called the complementary function) of the circuit, is the solution of the homogeneous differential equation:

$$L\frac{di_n}{dt} + Ri_n = 0. \tag{1.22}$$

The forced response (also called the particular integral) of the circuit, $i_f(t)$, is given by the solution of the nonhomoge-

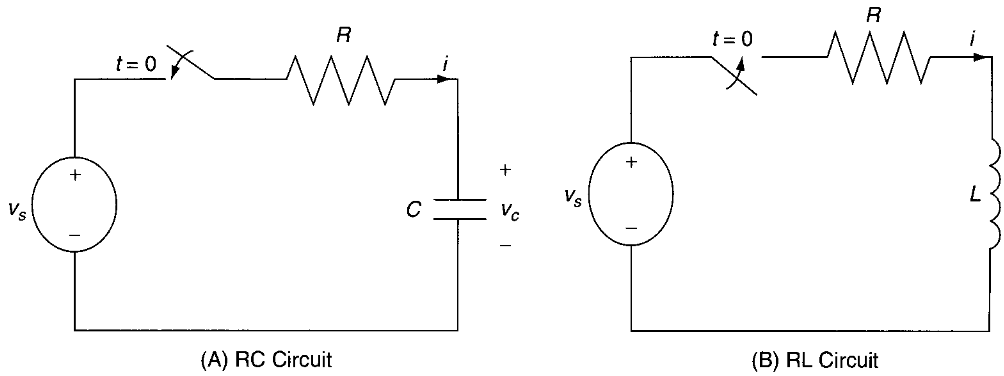


FIGURE 1.20 Circuits with a Single Energy-Storing Element

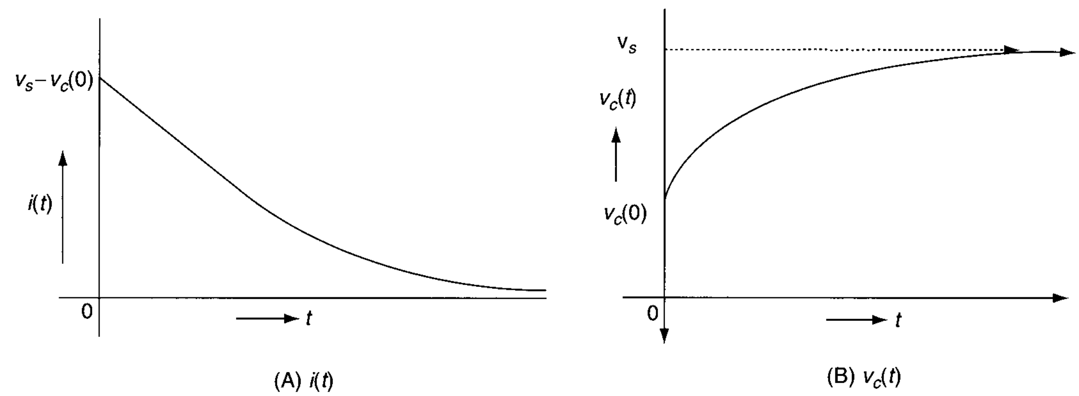


FIGURE 1.21 Response of the Circuit in Figure 1.20(A)

neous differential equation corresponding to the particular forcing function v_s. If v_s is a constant, the forced response in general is also a constant. In this case, the natural and forced responses and the total response are given by:

$$i_n(t) = Ke^{-R/L\,t},\ i_f(t) = \frac{v_s}{R},\ \text{and}\ i(t) = Ke^{-(R/L)t} + \frac{v_s}{R}. \tag{1.23}$$

K is found using the initial condition in the inductor $i(0) = I_0$ as $i(0) = K + v_s/R$, and so $K = I_0 - v_s/R$. Substituting for K in the total response yields:

$$i(t) = \left(I_0 - \frac{v_s}{R}\right)e^{-(R/L)t} + \frac{v_s}{R}. \tag{1.24}$$

The current waveform, shown in Figure 1.22, has an exponential characteristic with a time constant of L/R [s].

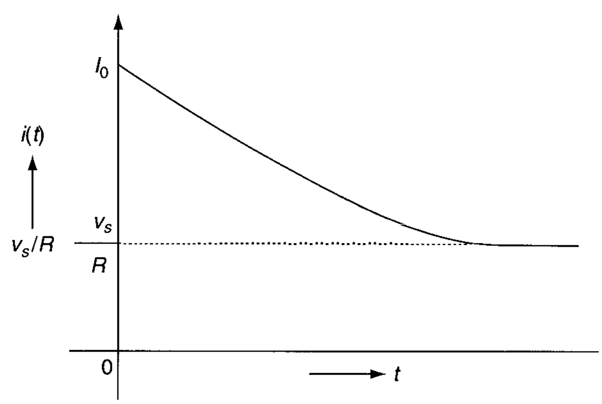


FIGURE 1.22 Response of the Circuit Shown in Figure 1.20(B)

1.6.2 Second-Order Circuits

If the circuit contains two energy-storing elements, L and/or C, the equation connecting voltage or current in the circuit is a second-order differential equation. Consider, for example, the circuit shown in Figure 1.23.

Writing KCL around the loop and substituting $i = Cdv_c/dt$ results in:

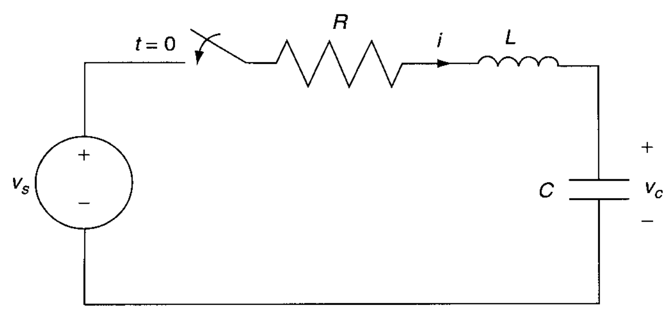


FIGURE 1.23 Circuit with Two Energy-Storing Elements

$$LC\frac{d^2v_c}{dt} + RC\frac{dv_c}{dt} + v_c = v_s. \qquad (1.25)$$

This equation can be solved by either using a Laplace transform or a conventional technique. This section illustrates the use of the conventional technique. Assuming a solution of the form $v_n(t) = Ke^{st}$ for the homogeneous equation yields the characteristic equation as:

$$LCs^2 + RCs + 1 = 0. \qquad (1.26)$$

Solving for s results in the characteristic roots written as:

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}. \qquad (1.27)$$

Four cases should be considered:

Case 1: $(R/2L)^2 > (1/LC)$. The result is two real negative roots s_1 and s_2 for which the solution will be an overdamped response of the form:

$$v_n(t) = K_1e^{s_1t} + K_2e^{s_2t}. \qquad (1.28)$$

Case 2: $(R/2L)^2 = (1/LC)$. In this case, the result is a double root at $s_0 = -R/2L$. The natural response is a critically damped response of the form:

$$v_n(t) = (K_1t + K_2)e^{s_0t}. \qquad (1.29)$$

Case 3: $0 < (R/2L)^2 < (1/LC)$. This case yields a pair of complex conjugate roots as:

$$s_{1,2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = -\sigma \pm j\omega_d. \qquad (1.30)$$

The corresponding natural response is an underdamped oscillatory response of the form:

$$v_n(t) = Ke^{-\sigma t}\cos(\omega_d t + \theta). \qquad (1.31)$$

Case 4: $R/2L = 0$. In this case, a pair of imaginary roots are created as:

$$s_{1,2} = \pm j\sqrt{\frac{1}{LC}} = \pm j\omega_0. \qquad (1.32)$$

The corresponding natural response is an undamped oscillation of the form:

$$v_n(t) = K\cos(\omega_0 t + \theta). \qquad (1.33)$$

The forced response can be obtained as $v_f(t) = V_s$. The total solution is obtained as:

$$v_c(t) = v_n(t) + V_s. \qquad (1.34)$$

The unknown coefficients K_1 and K_2 in cases 1 and 2 and K and θ in cases 3 and 4 can be calculated using the initial values on the current in the inductor and the voltage across the capacitor.

Typical responses for the four cases when V_s equals zero are shown in Figure 1.24. For circuits containing energy-dissipating elements, namely resistors, the natural response in general will die down to zero as t goes to infinity. The component of the response that goes to zero as time t goes to infinity is called the **transient response**. The forced response depends on the forcing function. When the forcing function is a constant or a sinusoidal function, the forced response will continue to be present even as t goes to infinity. The component of the total response that continues to exist for all time is called **steady state response**. In the next section, computation of steady state responses for sinusoidal forcing functions is considered.

1.6.3 Higher Order Circuits

When a circuit has more than two energy-storing elements, say n, the analysis of the circuit in general results in a differential equation of order n. The solution of such an equation follows steps similar to the second-order case. The characteristic equation will be of degree n and will have n roots. The natural response will have n exponential terms. Also, the forced response will in general have the same shape as the forcing function. The Laplace transform is normally used to solve such higher order circuits.

1.7 Laplace Transform

In the solution of linear time-invariant differential equations, it was noted that a forcing function of the form K_ie^{st} yields an output of the form K_oe^{st} where s is a complex variable. The function e^{st} is a complex sinusoid of exponentially varying amplitude, often called a damped sinusoid. Because linear equations obey the superposition principle, the solution of a linear differential equation to any forcing function can be found by superposing solutions to component-damped sinusoids if the forcing function is expressed as a sum of damped

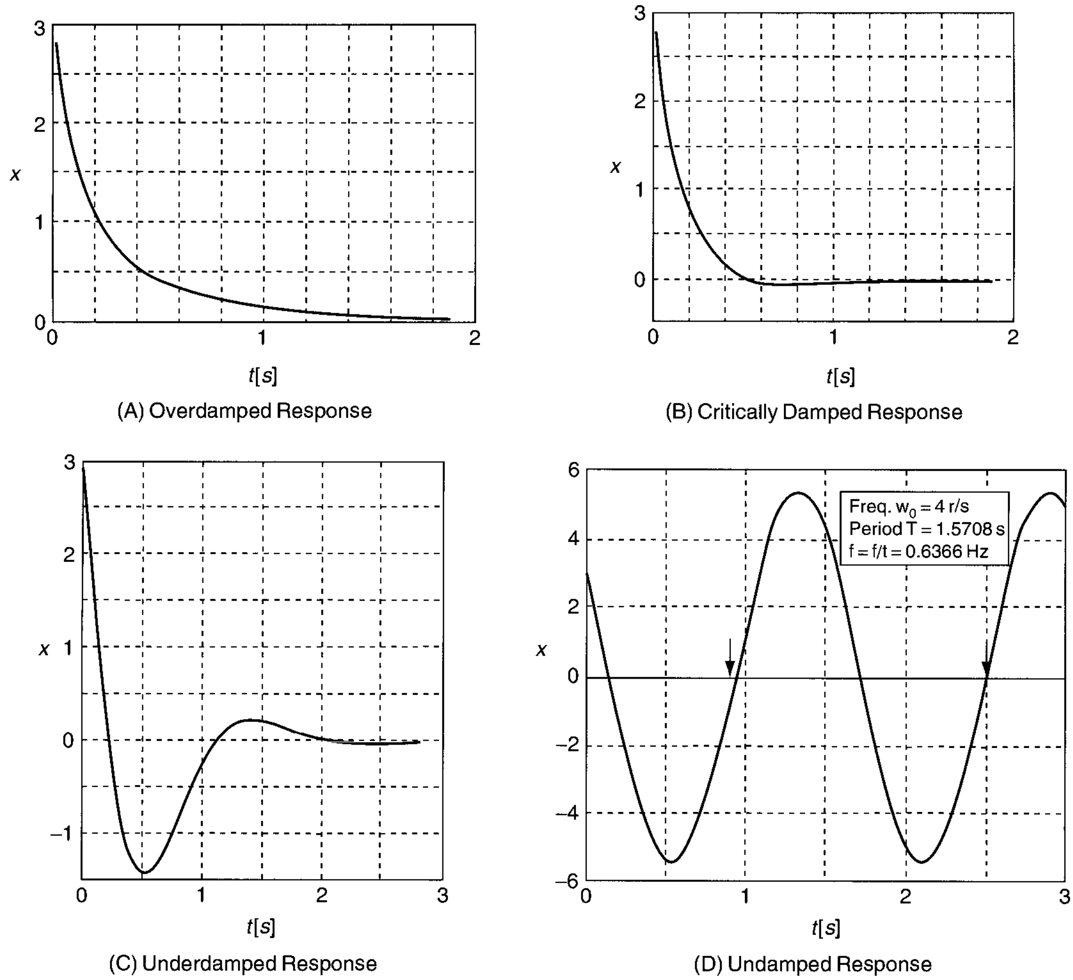


FIGURE 1.24 Typical Second-Order Circuit Responses

sinusoids. With this objective in mind, the Laplace transform is defined. The **Laplace transform** decomposes a given time function into an integral of complex-damped sinusoids.

1.7.1 Definition

The Laplace transform of $f(t)$ is defined as:

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt. \tag{1.35}$$

The inverse Laplace transform is defined as:

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_0-j\infty}^{\sigma_0+j\infty} F(s)e^{st}dt. \tag{1.36}$$

$F(s)$ is called the Laplace transform of $f(t)$, and σ_0 is included in the limits to ensure the convergence of the improper integral. The equation 1.36 shows that $f(t)$ is expressed as a sum (integral) of infinitely many exponential functions of complex frequencies (s) with complex amplitudes (phasors) $\{F(s)\}$. The complex amplitude $F(s)$ at any frequency s is given by the integral in equation 1.35. The Laplace transform, defined as the integral extending from zero to infinity, is called a single-sided Laplace transform against the double-sided Laplace transform whose integral extends from $-\infty$ to $+\infty$. As transient response calculations start from some initial time, the single-sided transforms are sufficient in the time domain analysis of linear electric circuits. Hence, this discussion considers only single-sided Laplace transforms.

1.7.2 Laplace Transforms of Common Functions

Consider

$$f(t) = Ae^{-at} \text{ for } 0 \leq t \leq \infty, \tag{1.37}$$

then

$$F(s) = \int_0^\infty Ae^{-at}e^{-st}dt = \left.\frac{Ae^{-(a+s)t}}{-(a+s)}\right|_0^\infty = \frac{Ae^{-\infty} - Ae^{-0}}{-(a+s)} = \frac{A}{s+a}. \quad (1.38)$$

In this equation, it is assumed that Re(s) – Re(a). In the region in the complex s-plane where s satisfies the condition that Res > Rea, the integral converges, and the region is called the **region of convergence of F(s)**. When $a = 0$ and $A = 1$, the above $f(t)$ becomes $u(t)$, the unit step function. Substituting these values in equation 1.38, the Laplace transform of $u(t)$ is obtained as $1/s$. In a similar way, letting $s = -j\omega$, the Laplace transform of $Ae^{j\omega t}$ is obtained as $A/(s - j\omega)$. Expressing $\cos(\omega t) = (e^{j\omega t} + e^{-j\omega t})/2$, we get the Laplace transform of $A\cos(\omega t)$ as $As/(s^2 + \omega^2)$. In a similar way, the Laplace transform of $A\sin(\omega t)$ is obtained as $A\omega/(s^2 + w^2)$. Transforms for some commonly occurring functions are given in Table 1.4. This table can be used for finding forward as well as inverse transforms of functions.

As mentioned at the beginning of this section, the Laplace transform can be used to solve linear time-invariant differential equations. This will be illustrated next in example 1.7.

EXAMPLE 1.7. *Consider the second-order differential equation and use the Laplace transform to find a solution:*

$$\frac{d^2f}{dt^2} + 6\frac{df}{dt} + 8f = 4e^{-t} \quad (1.39)$$

with initial conditions $f(0) = 2$ and $\frac{df}{dt}(0) = 3$.

Taking the Laplace transform of both sides of the above differential equation produces:

TABLE 1.4 Laplace Transforms of Common Functions

$f(t)$, for $t \geq 0$	$F(s)$
A	$\frac{A}{s}$
$Ae^{-\sigma t}$	$\frac{A}{s+\sigma}$
At	$\frac{A}{s^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-\sigma t}\cos(\omega t)$	$\frac{s+\sigma}{(s+\sigma)^2+\omega^2}$
$e^{-\sigma t}\sin(\omega t)$	$\frac{\omega}{(s+\sigma)^2+\omega^2}$

$$s^2F(s) - sf(0) - \frac{df}{dt}(0) + 6(sF(s) - f(0)) + 8F(s) = \frac{4}{(s+1)}. \quad (1.40)$$

Substituting for the initial values, you get:

$$s^2F(s) - 2s - 3 + 6sF(s) - 12 + 8F(s) = \frac{4}{(s+1)}. \quad (1.41)$$

$$(s^2 + 6s + 8)F(s) = \frac{(2s+15)(s+1)+4}{(s+1)} = \frac{(2s^2+17s+19)}{(s+1)}. \quad (1.42)$$

$$F(s) = \frac{(2s^2+17s+19)}{(s^2+6s+8)(s+1)} = \frac{(2s^2+17s+19)}{(s+2)(s+4)(s+1)}. \quad (1.43)$$

Applying partial fraction expansion, you get:

$$F(s) = \frac{7/2}{s+2} + \frac{-17/6}{s+4} + \frac{4/3}{s+1}. \quad (1.44)$$

Taking the inverse Laplace transform using the Table 1.5, you get:

$$f(t) = \frac{-7e^{-2t}}{2}\frac{-17e^{-4t}}{6} + \frac{4e^{-t}}{3} \text{ for } t > 0. \quad (1.45)$$

It may be noted that the total solution is obtained in a single step while taking the initial conditions along the way.

1.7.3 Solution of Electrical Circuits Using the Laplace Transform

There are two ways to apply the Laplace transform for the solution of electrical circuits. In one method, the differential equations for the circuit are first obtained, and then the differential equations are solved using the Laplace transform. In the second method, the circuit elements are converted into s-domain functions and KCL and KVL are applied to the s-domain circuit to obtain the needed current or voltage in the s-domain. The current or voltage in time domain is obtained using the inverse Laplace transform. The second method is simpler and is illustrated here.

Let the Laplace transform of $\{v(t)\} = V(s)$ and Laplace transform of $\{i(t)\} = I(s)$. Then the s-domain voltage current relations of the R, L, and C elements are obtained as follows. Consider a resistor with the v–i relation:

$$v(t) = R\,i(t). \quad (1.46)$$

Taking the Laplace transform on both the sides yields:

$$V(s) = R\,I(s). \quad (1.47)$$

TABLE 1.5 Properties of Laplace Transforms

Operations	$f(t)$	$F(s)$
Addition	$f_1(t)+f_2(t)$	$F_1(s)+F_2(s)$
Scalar multiplication	$Af(t)$	$AF(s)$
Time differentiation	$d/dt\{f(t)\}$	$sF(s)-f(0)$
Time integration	$\int_0^\infty f(t)dt$	$\frac{F(s)}{s}$
Convolution	$\int_0^\infty f_1(t-\tau)f_2(\tau)d\tau$	$F_1(s)F_2(s)$
Frequency shift	$f(t)e^{-at}$	$F(s+a)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency differentiation	$-tf(t)$	$\frac{dF}{ds}$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)ds$
Scaling	$f(at),\ a>0$	$\frac{1}{a}F(\frac{s}{a})$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$

Note: The $u(t)$ is the unit step function defined by $u(t)=0$ for $t<0$ and $u(t)=1$ for $t>0$.

Defining the impedance of an element as $V(s)/I(s)=Z(s)$ produces $Z(s)=R$ for a resistance. For an inductance, $v(t)=L\,di/dt$. Taking the Laplace transform of the relation yields $V(s)=sLI(s)-Li(0)$, where $i(0)$ represents the initial current in the inductor and where $Z(s)=sL$ is the impedance of the inductance. For a capacitance, $i(t)=c\,dv/dt$ and $I(s)=sc\,V(s)-cv(0)$, where $v(0)$ represents the initial voltage across the capacitance and where $1/sc$ is the impedance of the capacitance.

Equivalent circuits that correspond to the s-domain relations for R, L, and C are shown in Table 1.6 and are suitable for writing KVL equations (initial condition as a voltage source) as well as for writing KCL equations (initial condition as a current source). With these equivalent circuits, a linear circuit can be converted to an s-domain circuit as shown in the example 1.8.

It is important first to show that the KCL and KVL relations can also be converted into s-domain relations. For example, the KCL relation in s-domain is obtained as follows: At any node, KCL states that:

$$i_1(t)+i_2(t)+i_3(t)+\ldots+i_n(t)=0. \quad (1.48)$$

By applying Laplace transform on both sides, the result is:

$$I_1(s)+I_2(s)+I_3(s)+\ldots+I_n(s)=0, \quad (1.49)$$

which is the KCL relation for s-domain currents in a node. In a similar manner, the KVL around a loop can be written in s-domain as:

$$V_1(s)+V_2(s)+\ldots+V_n(s)=0, \quad (1.50)$$

where $V_1(s)$, $V_2(s)$, ..., $V_n(s)$ are the s-domain voltages around the loop. In fact, the various time-domain theorems considered earlier, such as the superposition, Thevenin, and Norton theorems, series and parallel equivalent circuits and voltage and current divisions are also valid in the s-domain. The loop current method and node voltage method can be applied for analysis in s-domain.

EXAMPLE 1.8. *Consider the circuit given in Figure 1.25(A) and convert a linear circuit into an s-domain circuit.* You can obtain the s-domain circuit shown in Figure 1.25(B) by replacing each element by its equivalent s-domain element. As noted previously, the differential relations of the elements on application of the Laplace transform have become algebraic relations. Applying KVL around the loop, you can obtain the following equations:

$$2I(s)+0.2sI(s)-0.4+\frac{6}{s}I(s)+\frac{3}{s}=\frac{10}{s}. \quad (1.51)$$

$$(2+0.2s+\frac{6}{s})I(s)=\frac{10}{s}-\frac{3}{s}+0.4=\frac{7+0.4s}{s}. \quad (1.52)$$

$$I(s)=\frac{2s+35}{s^2+10s+30}=\frac{2(s+5)+25}{(s+5)^2+(\sqrt{5})^2}. \quad (1.53)$$

$$i(t)=e^{-5t}\{2\cos(\sqrt{5}t)+5\sqrt{5}\sin(\sqrt{5}t)\} \text{ for } t>0. \quad (1.54)$$

From 1.51 to 1.54 equations, you can see that the solution using the Laplace transform determines the natural response and forced response at the same time. In addition, the initial conditions are taken into account when the s-domain circuit is set up. A limitation of the Laplace transform is that it can be used only for linear circuits.

1.7.4 Network Functions

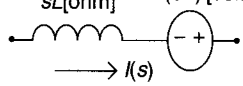
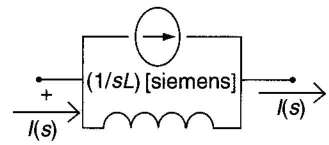
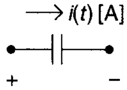
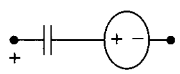
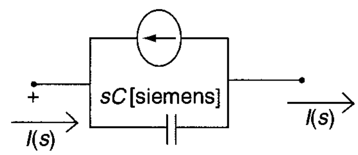
For a one-port network, voltage and current are the two variables associated with the input port, also called the driving port. One can define two driving point functions under zero initial conditions as:

$$\text{Driving point impedance } Z(s)=\frac{V(s)}{I(s)}.$$

$$\text{Driving point admittance } Y(s)=\frac{I(s)}{V(s)}.$$

In the case of two-port networks, one of the ports may be considered as an input port with input signal $X(s)$ and the other considered the output port with output signal $Y(s)$. Then the transfer function is defined as:

TABLE 1.6 *s*-Domain Equivalent Circuits for *R*, *I*, and *C* Elements

Time domain	Laplace domain KVL	Laplace domain KCL
R [ohm] $i(t)$ [ampere]  $v(t)$ [volt] $v(t) = R\,i(t)$	R [ohm] → $I(s)$ [ampere]  $V(s)$ [volt] $V(s) = RI(s)$	G [siemens] → $I(s)$ [ampere]  $V(s)$ [volt] $I(s) = G\,V(s)$
L[henry] → $i(t)$ [ampere]  $v(t)$ [volt] $v(t) = Ldi/dt$	sL[ohm] $Li(0+)$ [volt]  → $I(s)$ + ← $V(s)$ → $V(s) = (sL)\,I(s) - Li(0+)$	$i(0+)/s$  (1/sL) [siemens] $I(s)$ $I(s)$ + ← $V(s)$ → $I(s) = (1/sL)\,V(s) + [i(0+)/s]$
C[farad] → $i(t)$ [A]  $v(t)$ [volt] $i(t) = C\,dv/dt$	(1/sC)[ohm] ($v(0+)/s$)  → I(s) ← V(s) → $V(s) = (1/sC)I(s) + (v(0+)/s)$	$Cv(0+)$ [ampere]  sC [siemens] $I(s)$ $I(s)$ + ← $V(s)$ → $I(s) = (sC)\,V(s) - Cv(0+)$

Note: [A] represents ampere, and [*V*] represents volt.

$$H(s) = \frac{Y(s)}{X(s)}, \text{ under zero initial conditions.}$$

In an electrical network, both $Y(s)$ and $X(s)$ can be either voltage or current variables. Four transfer functions can be defined as:

Transfer voltage ratio $G_v(s) = \dfrac{V_2(s)}{V_1(s)}$,
under the condition $I_2(s) = 0$.

Transfer current ratio $G_i(s) = \dfrac{I_2(s)}{I_1(s)}$,
under the condition $V_2(s) = 0$.

Transfer impedance $Z_{21} = \dfrac{V_2(s)}{I_1(s)}$,
under the condition $I_2(s) = 0$.

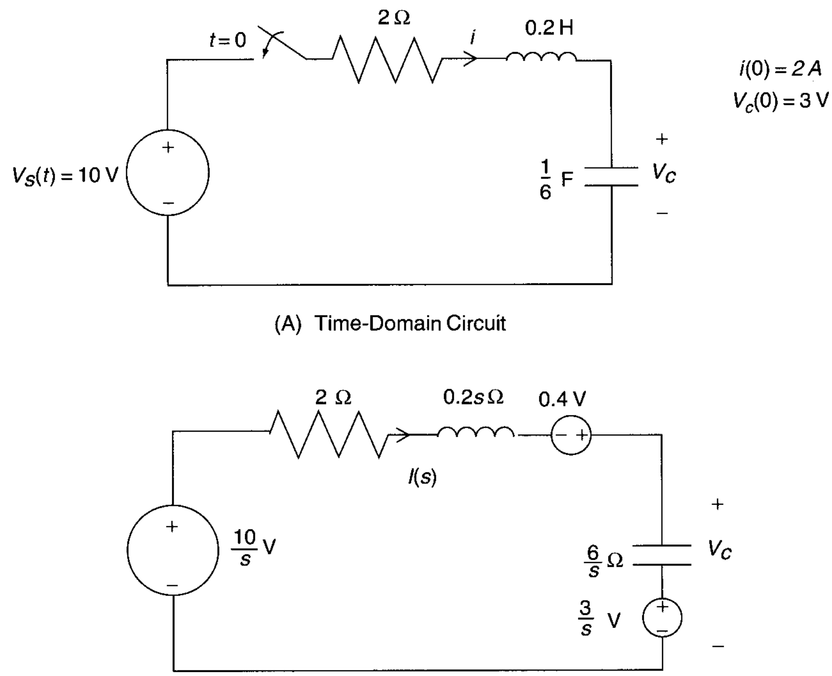
Transfer admittance $Y_{21} = \dfrac{I_2(s)}{V_1(s)}$,
under the condition $V_2(s) = 0$.

1.8 State Variable Analysis

State variable analysis or state space analysis, as it is sometimes called, is a matrix-based approach that is used for analysis of circuits containing time-varying elements as well as nonlinear elements. The state of a circuit or a system is defined as a set of a minimum number of variables associated with the circuit; knowledge of these variables along with the knowledge of the input will enable the prediction of the currents and voltages in all system elements at any future time.

1.8.1 State Variables for Electrical Circuits

As was mentioned earlier, only capacitors and inductors are capable of storing energy in a circuit, and so only the variables associated with them are able to influence the future condition of the circuit. The voltages across the capacitors and the currents through the inductors may serve as state variables. If loops are not solely made up of capacitors and voltage sources, then the voltages across all the capacitors are independent



(A) Time-Domain Circuit

(B) s-Domain Equivalent Circuit

FIGURE 1.25 Circuit for Example 1.8

variables and may be taken as state variables. In a similar way, if there are no sets of inductors and current sources that separate the circuit into two or more parts, then the currents associated with the inductors are independent variables and may be taken as state variables. The following examples assume that all the capacitor voltages and inductor currents are independent variables and will form the set of state variables for the circuit.

1.8.2 Matrix Representation of State Variable Equations

Because matrix techniques are used in state variable analysis, the state variables are commonly expressed as a vector $\boldsymbol{x}$, and the input source variables are expressed as a vector $\boldsymbol{r}$. The output variables are denoted as $\boldsymbol{y}$.

Once the state variables $\boldsymbol{x}$ are chosen, KVL and KCL are used to determine the derivatives $\dot{x}$ of the state variables and the output variables y in terms of the state variables $\boldsymbol{x}$ and source variables $\boldsymbol{r}$. They are expressed as:

$$\dot{x} = Ax + Br.$$
$$y = Cx + Dr. \qquad (1.55)$$

The $\dot{x}$ equation is called the **state dynamics equation**, and the y equation is called the **output equation**. A, B, C, and D are appropriately dimensioned coefficient matrices. This set of equations, where the derivative of state variables is expressed as a linear combination of state variables and forcing functions, is said to be in **normal form**.

Example 1.9. *Consider the circuit shown in Figure 1.26 and solve for state variable equations in matrix form.*

Taking v_L and i_C as state variables and applying KVL around loop ABDA, you get:

$$\frac{di_L}{dt} = \frac{1}{L}v_L = \frac{1}{L}(v_a - v_C) = \frac{1}{L}v_a - \frac{1}{L}v_C. \qquad (1.56)$$

Similarly, by applying KCL at node B, you get:

$$\frac{dv_C}{dt} = \frac{1}{C}(i_C) = \frac{1}{C}(i_L + i_1). \qquad (1.57)$$

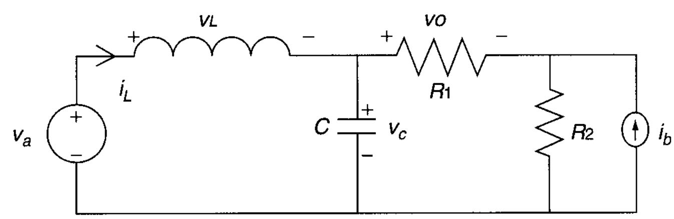


FIGURE 1.26 Circuit for Example 1.9

The current i_1 can be found either by writing node equation at node C or by applying the superposition as:

$$i_1 = \frac{R_2}{R_1 + R_2} i_b - \frac{1}{R_1 + R_2} v_c. \tag{1.58}$$

Substituting for i_1 in equation 1.57, you get:

$$\frac{dv_C}{dt} = \frac{1}{C} i_L + \frac{1}{C(R_1 + R_2)} i_b - \frac{1}{C(R_1 + R_2)} v_c. \tag{1.59}$$

The output v_o can be obtained as $-i_1 R_1$ and can be expressed in terms of state variables and sources by employing equation 1.58 as:

$$v_o = -\frac{R_1 R_2}{R_1 + R_2} i_b + \frac{R_1}{R_1 + R_2} v_c. \tag{1.60}$$

Equations 1.56, 1.59, and 1.60 can be expressed in matrix form as:

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & \frac{1}{C(R_1 + R_2)} \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{C(R_1 + R_2)} \end{bmatrix} \begin{bmatrix} v_a \\ i_b \end{bmatrix}. \tag{1.61}$$

$$[v_o] = \begin{bmatrix} 0 & \frac{R_1}{R_1 + R_2} \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 & -\frac{R_1 R_2}{R_1 + R_2} \end{bmatrix} \begin{bmatrix} v_a \\ i_b \end{bmatrix}. \tag{1.62}$$

The ordering of the variables in the state variable vector $\boldsymbol{x}$ and the input vector $\boldsymbol{r}$ is arbitrary. Once the ordering is chosen, however, it must remain the same for all the variables in every place of occurrence. In large circuits, topological methods may be employed to systematically select the state variables and write KCL and KVL equations. For want of space, these methods are not described in this discussion. Next, this chapter briefly goes over the method for solving state variable equations.

1.8.3 Solution of State Variable Equations

There are many methods for solving state variable equations: (1) computer-based numerical solution, (2) conventional differential equation time-domain solution, and (3) Laplace transform s-domain solution. This chapter will only present the Laplace transform domain solution.

Consider the state dynamics equation:

$$\dot{x} = Ax + Br. \tag{1.63}$$

Taking the Laplace transform on both sides yields:

$$sX(s) - x(0) = AX(s) + BR(s), \tag{1.64}$$

where $X(s)$ and $R(s)$ are the Laplace transforms of $x(t)$ and $r(t)$, respectively, and $x(0)$ represents the initial conditions.

Rearranging equation 1.64 results in:

$$\begin{aligned} (sI - A)X(s) &= x(0) + BR(s) \\ X(s) &= (sI - A)^{-1}x(0) + (sI - A)^{-1}BR(s). \end{aligned} \tag{1.65}$$

Taking the inverse Laplace transform of $X(s)$ yields:

$$x(t) = \phi(t)x(0) + \phi(t) * Br(t), \tag{1.66}$$

where $\phi(t)$, the inverse Laplace transform of $\{(sI - A)^{-1}\}$, is called the state transition matrix and where $*$ represents the time domain convolution.

Expanding the convolution, $x(t)$ can be written as:

$$x(t) = \phi(t)x(0) + \int_0^t \phi(t - \tau)Br(\tau)d\tau. \tag{1.67}$$

Once $x(t)$ is known, $y(t)$ may be found using the output equation 1.60.

1.9 Alternating Current Steady State Analysis

1.9.1 Sinusoidal Voltages and Currents

Standard forms of writing sinusoidal voltages and currents are:

$$v(t) = V_m \cos(\omega t + \alpha)[\mathrm{V}]. \tag{1.68a}$$

$$i(t) = I_m \cos(\omega t + \beta)[\mathrm{A}]. \tag{1.68b}$$

V_m and I_m are the maximum values of the voltage and current, ω is the frequency of the signal in radians/second, and α and β are called the phase angles of the voltage and current, respectively. V_m, I_m, and ω are positive real values, whereas α and β are real and can be positive or negative. If α is greater than β, the voltage is said to lead the current, or the current to lag the voltage. If α is less than β the voltage is said to lag the current, or the current to lead the voltage. If α equals β, the voltage and current are in phase.

1.9.2 Complex Exponential Function

Define $X_m e^{j(\omega t+\phi)}$ as a complex exponential function. By Euler's theorem,

$$e^{j\theta} = \cos\theta + j\sin\theta.$$
$$X_m e^{j(\omega t+\phi)} = X_m[\cos(\omega t+\phi) + j\sin(\omega t+\phi)].$$
$$x(t) = X_m \cos(\omega t+\phi).$$
$$= \text{Real}\ [X_m e^{j(\omega t+\phi)}] = \text{Real}\ [(X_m e^{j\phi})e^{j\omega t}]. \quad (1.69)$$

The term $(X_m e^{j\phi})$ is called the phasor of the sinusoidal function $x(t)$. For linear RLCM circuits, the forced response is sinusoidal at the input frequency. Since the natural response decays exponentially in time, the forced response is also the steady state response.

1.9.3 Phasors in Alternating Current Circuit Analysis

Consider voltage and current waves of the same frequency:

$$v(t) = V_m \cos(\omega t+\alpha)[\text{V}].$$
$$i(t) = I_m \cos(\omega t+\beta)[\text{A}].$$

Alternative representation is by complex exponentials:

$$v(t) = \text{Real}\ [V_m e^{j\alpha}]e^{j\omega t}[\text{V}].$$
$$i(t) = \text{Real}\ [I_m e^{j\beta}]e^{j\omega t}[\text{A}].$$

Phasor voltage and current are defined as:

$$\boldsymbol{V} = V_m e^{j\alpha}[\text{V}]. \quad (1.70a)$$
$$\boldsymbol{I} = I_m e^{j\beta}[\text{A}]. \quad (1.70b)$$

Since a phasor is a complex number, other representations of a complex number can be used to specify the phasor. These are listed in Table 1.7.

Addition of two voltages or two currents of the same frequency is accomplished by adding the corresponding phasors.

1.9.4 Phasor Diagrams

Since a phasor is a complex number, it can be represented on a complex plane as a *vector* in Cartesian coordinates. The length of the vector is the magnitude of the phasor, and the direction is the phase angle. The projection of the vector on the x-axis is the real part of the phasor, and the projection on the y-axis is the imaginary part of the phasor in rectangular form as noted in Figure 1.27. The graphical representation is called the phasor diagram and the vector is called the phasor.

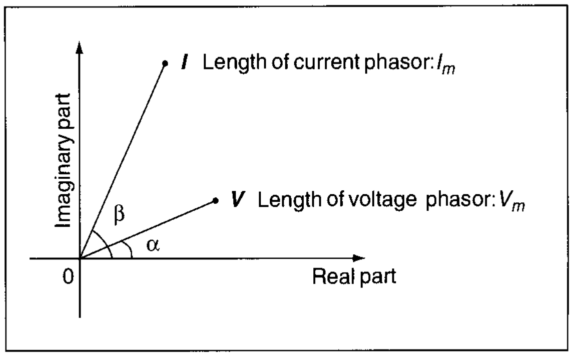


FIGURE 1.27 Phasor Diagram

TABLE 1.7 Representation of Phasor Voltages and Currents

Phasor	Exponential form	Rectangular form	Polar form
$\boldsymbol{V}$	$V_m e j\alpha$	$V_m\cos\alpha + jV_m\sin\alpha$	$V_m\angle\alpha$[V]
$\boldsymbol{I}$	$I_m e j\beta$	$I_m\cos\beta + jI_m\sin\beta$	$I_m\angle\beta$[A]

1.9.5 Phasor Voltage–Current Relationships of Circuit Elements

Voltage $v(t)$ and current $i(t)$ are sinusoidal signals at a frequency of ω rad/s, whereas $\boldsymbol{V}$ and $\boldsymbol{I}$ are phasor voltage and current, respectively. The v–i and **V**–**I** relations for R, L, and C elements are given in Table 1.8.

1.9.6 Impedances and Admittances in Alternating Current Circuits

Impedance $\boldsymbol{Z}$ is defined as the ratio of phasor voltage to phasor current at a pair of terminals in a circuit. The unit of impedance is ohms. **Admittance** $\boldsymbol{Y}$ is defined as the ratio of phasor current to phasor voltage at a pair of terminals of a circuit. The unit of admittance is siemens. $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are complex numbers and reciprocals of each other. Note that phasors are also complex numbers, but phasors represent time-varying sinusoids. Impedance and admittance are time invariant and frequency dependent. Table 1.9 shows the impedances and admittances for R, L, and C elements. It may be noted that the phasor impedance for an element is obtained by substituting $s = j\omega$ in the s-domain impedance $Z(s)$ of the corresponding element.

TABLE 1.8 Element Voltage–Current Relationships

Element	Time domain	Frequency domain
Resistance: R	$v = Ri$	$\boldsymbol{V} = R\boldsymbol{I}$
Inductance: L	$v = Ldi/dt$	$\boldsymbol{V} = j\omega L\boldsymbol{I}$
Capacitance: L	$i = Cdv/dt$	$\boldsymbol{I} = j\omega C\boldsymbol{V}$

TABLE 1.9 Impedances and Admittances of Circuit Elements

Element	Resistance: R	Inductor: L	Capacitor: C
Z [ohm]	R	$j\,X_L$ $X_L = \omega L$ (X_L is the inductive reactance.)	$j\,X_C$ $X_C = -1/\omega\, C$ (X_C is the capacitive reactance.)
Y [siemens]	$G = 1/R$ (G is the conductance.)	$j\,B_L$ $B_L = -1/\omega\, L$ (B_L is the inductive susceptance.)	$j\,B_C$ $B_C = \omega C$ (B_C is the capacitive susceptance.)

1.9.7 Series Impedances and Parallel Admittances

If n impedances are in series, their equivalent impedance is given by $Z_{eq} = (Z_1 + Z_2 + \ldots + Z_n)$. Similarly, the equivalent admittance of n admittances in parallel is given by $Y_{eq} = (Y_1 + Y_2 + \ldots + Y_n)$.

1.9.8 Alternating Current Circuit Analysis

Before the steady state analysis is made, a given time-domain circuit is replaced by its phasor-domain circuit, also called the **phasor circuit**. The phasor circuit of a given time-domain circuit at a specified frequency is obtained as follows:

- Voltages and currents are replaced by their corresponding phasors.
- Circuit elements are replaced by impedances or admittances at the specified frequency given in Table 1.9.

All circuit analysis techniques are now applicable to the phasor circuit.

1.9.9 Steps in the Analysis of Phasor Circuits

- Select mesh or nodal analysis for solving the phasor circuit.
- Mark phasor mesh currents or phasor nodal voltages.
- Use impedances for mesh analysis and admittances for nodal analysis.
- Write KVL around meshes (loops) or KCL at the nodes.
  KVL around a mesh: The algebraic sum of phasor voltage drops around a mesh is zero.
  KCL at a node: The algebraic sum of phasor currents leaving a node is zero.
- Solve the mesh or nodal equations with complex coefficients and obtain the complex phasor mesh currents or nodal voltages. The solution can be obtained by variable elimination or Cramer's rule. Remember that the arithmetic is complex number arithmetic.

1.9.10 Methods of Alternating Current Circuit Analysis

All methods of circuit analysis are applicable to alternating current (ac) phasor circuits. Phasor voltages and currents are the variables. Impedances and admittances describe the element voltage–current relationships. Some of the methods are described in this section.

Method of superposition: Circuits with multiple sources of the same frequency can be solved by using the mesh or nodal analysis method on the phasor circuit at the source frequency. Alternatively, the principle of superposition in linear circuits can be applied. First, solve the phasor circuit for each independent source separately. Then add the response voltages and currents from each source to get the total response. Since the responses are at the same frequency, phasor addition is valid.

Voltage and current source equivalence in ac circuits: An ac voltage source in series with an impedance can be replaced across the same terminals by an equivalent ac current source of the same frequency in parallel with an admittance, as shown in Figure 1.28. Similarly, an ac current source in parallel with an admittance can be replaced across the same terminals by an equivalent ac voltage source of the same frequency in series with an impedance.

Current and voltage division in ac circuits: For two impedances in series:

$$V_1 = \left(\frac{Z_1}{Z_1 + Z_2}\right)V, \quad V_2 = \left(\frac{Z_2}{Z_1 + Z_2}\right)V$$

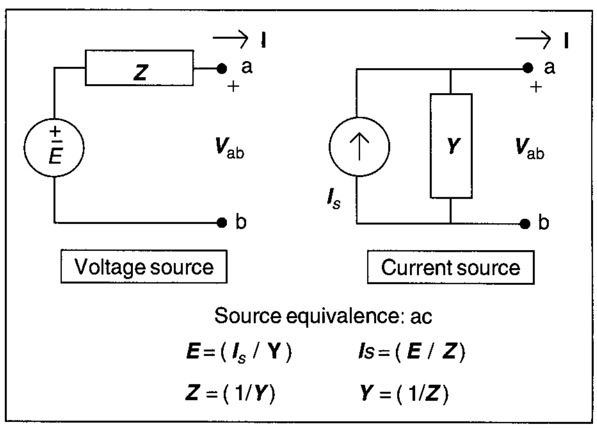


FIGURE 1.28 Source Transformations

For n impedances in series

$$V_1 = \left(\frac{Z_1}{\sum_{i=1}^{n} Z_i}\right) V, \quad V_2 = \left(\frac{Z_2}{\sum_{i=1}^{n} Z_i}\right) V, \ldots, V_n = \left(\frac{Z_n}{\sum_{i=1}^{n} Z_i}\right) V.$$

For two admittances in parallel

$$I_1 = \left(\frac{Y_1}{Y_1 + Y_2}\right) I, \quad I_2 = \left(\frac{Y_2}{Y_1 + Y_2}\right) I$$

For n admittances in parallel:

$$I_1 = \left(\frac{Y_1}{\sum_{i=1}^{n} Y_i}\right) I, \quad I_2 = \left(\frac{Y_2}{\sum_{i=1}^{n} Y_i}\right) I, \ldots, I_n = \left(\frac{Y_n}{\sum_{i=1}^{n} Y_i}\right) I.$$

Thevenin theorem for ac circuits: A phasor circuit across a pair of terminals is equivalent to an ideal phasor voltage source V_{oc} in series with an impedance Z_{Th}, where V_{oc} is the open circuit voltage across the terminals and where Z_{Th} is the equivalent impedance of the circuit across the specified terminals.

Norton theorem for ac circuits: A phasor circuit across a pair of terminals is equivalent to an ideal phasor current source I_{sc} in parallel with an admittance Y_N, where I_{sc} is the short circuit current across the terminals and where Y_N is the equivalent admittance of the circuit across the terminals.

Thevenin and Norton equivalent circuits of a linear phasor circuit are shown in Figure 1.29.

1.9.11 Frequency Response Characteristics

The voltage gain G_v at a frequency ω of a two-port network is defined as the ratio of the output voltage phasor to the input voltage phasor. In a similar manner, the current gain G_i is defined as the ratio of output current phasor to input current phasor. Because the phasors are complex quantities that depend on frequency, the gains, voltages, and currents are written as $G(j\omega)$, $V(j\omega)$, and $I(j\omega)$. Frequency response is then defined as the variation of the gain as a function of frequency. The above gain functions are also called transfer functions and written as $H(j\omega)$. $H(j\omega)$ is a complex number and can be written in polar form as follows:

$$H(j\omega) = A(\omega)/\phi(\omega) \tag{1.71}$$

where $A(\omega)$ is the gain magnitude function, $|H(j\omega)|$, and $\phi(\omega)$ is the phase function given by the argument of $H(j\omega)$. In addition to the above gain functions, a number of other useful ratios (network functions) among the voltages and currents of two port networks can be defined. These definitions and their nomenclature are given in Figure 1.30.

1.9.12 Bode Diagrams

Bode diagrams are graphical representations of the frequency responses and are used in solving design problems. Magnitude and phase functions are shown on separate graphs using logarithmic frequency scale along the x-axis. Logarithm of the frequency to base 10 is used for the x-axis of a graph. Zero frequency will correspond to negative infinity on the logarithmic scale and will not show on the plots. The x-axis is graduated in $\log_{10}\omega$ and so every decade of frequency (e.g.,...0.001, 0.01, 0.1, 1, 10, 100,...) is equally spaced on the x-axis.

The gain magnitude, represented by decibels defined as $20 \log_{10}[A(\omega)]$, is plotted on the y-axis of magnitude plot. Since $A(\omega)$ dB can be both positive and negative, the y-axis has both positive and negative values. Zero dB corresponds to a magnitude function of unity. The y-axis for the phase function uses a linear scale in radians or degrees. Semilog graph paper makes it convenient to sketch Bode plots.

Bode plots are easily sketched by making asymptotic approximations first. The frequency response function $H(j\omega)$ is a rational function, and the numerator and denominator are factorized into first-order terms and second-order terms with complex roots. The factors are then written in the standard form as follows:

$$\text{First-order terms: } (j\omega + \omega_0) \rightarrow \omega_0\left(1 + j\frac{\omega}{\omega_0}\right).$$

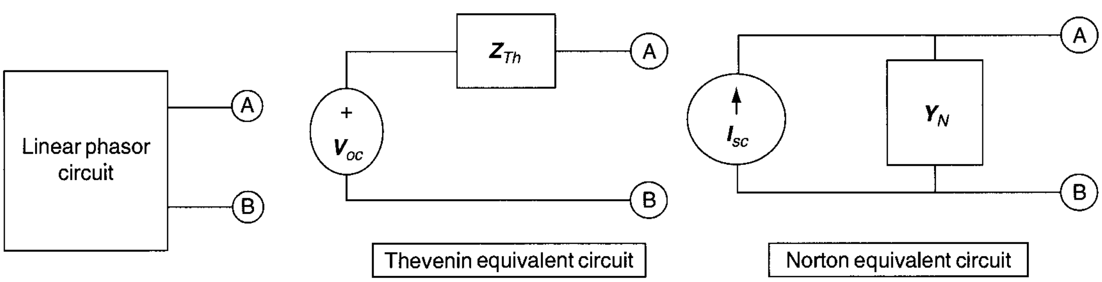


FIGURE 1.29 Thevenin and Norton Equivalent Circuits

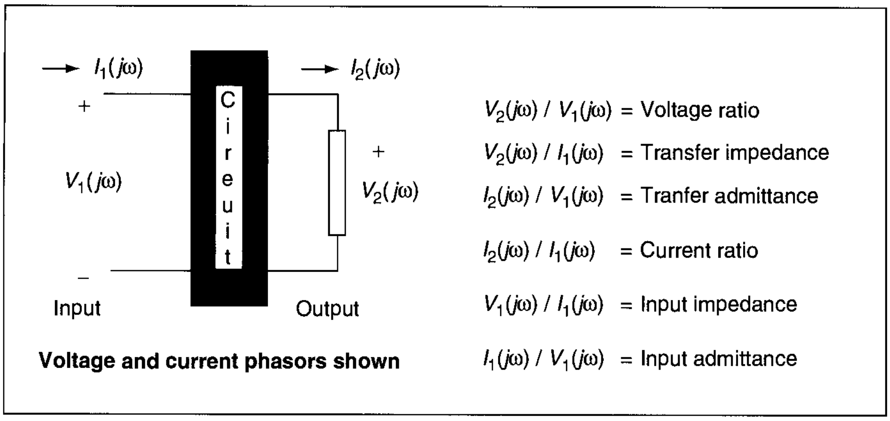


FIGURE 1.30 Frequency Response Functions
Notes: The frequency response ratios are called network functions.
When the input and output are at different terminals, frequency response function is also called a transfer function.

Second-order terms: $[(\omega_0^2 - \omega^2) + j(2\zeta\omega_0\omega)]$

$$\rightarrow \omega_0^2\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)+j\left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)\right].$$

Here ζ is the damping ratio with a value of less than 1. The magnitude and phase Bode diagrams are drawn by adding the individual plots of the first- and second-order terms. Asymptotic plots are first easily sketched by using approximations. For making asymptotic Bode plots, the ratio ω/ω_0 is assumed to be much smaller than one or much larger than one, so the Bode plots become straight line segments called asymptotic approximations. The normalizing frequency ω_0 is called the corner frequency. The asymptotic approximations are corrected at the corner frequencies by calculating the exact value of magnitude and phase functions at the corner frequencies.

The first- and second-order terms can occur in the numerator or denominator of the rational function $H(j\omega)$. Normalized plots for these terms are shown in figures 1.31 and 1.32. Bode diagrams for any function can be made using the normalized plots as building blocks. Figure 1.31 shows the Bode diagrams for a first-order term based on the following equations:

$$\textbf{(i)}\ H(j\omega) = \left(1 + j\frac{\omega}{\omega_0}\right). \quad \textbf{(ii)}\ H(j\omega) = \left(1 + j\frac{\omega}{\omega_0}\right)^{-1}.$$

The magnitude and phase plots of a denominator second-order term with complex roots are given in Figure 1.32. If the term is in the numerator, the figures are flipped about the x-axis, and the sign of the y-axis calibration is reversed.

1.10 Alternating Current Steady State Power

1.10.1 Power and Energy

Power is the rate at which energy E is transferred, such as in this equation:

$$p(t) = \frac{dE}{dt}$$

Energy is measured in Joules [J]. A unit of power is measured in watts [W]: $1\text{W} = 1\text{J/s}$. When energy is transferred at a constant rate, power is constant. In general, energy transferred over time t is the integral of power.

$$E = \int_0^T p(t)dt$$

1.10.2 Power in Electrical Circuits

Power in a two-terminal circuit at any instant is obtained by multiplying the voltage across the terminals by the current through the terminals:

$$p(t) = v(t)i(t). \tag{1.72}$$

If the voltage is in volts [V] and the current in amperes [A], power is in watts. In direct current (dc) circuits under steady state, the voltage $V$and current I are constant and power is also constant, given by $P = VI$. In ac circuits under steady state, V

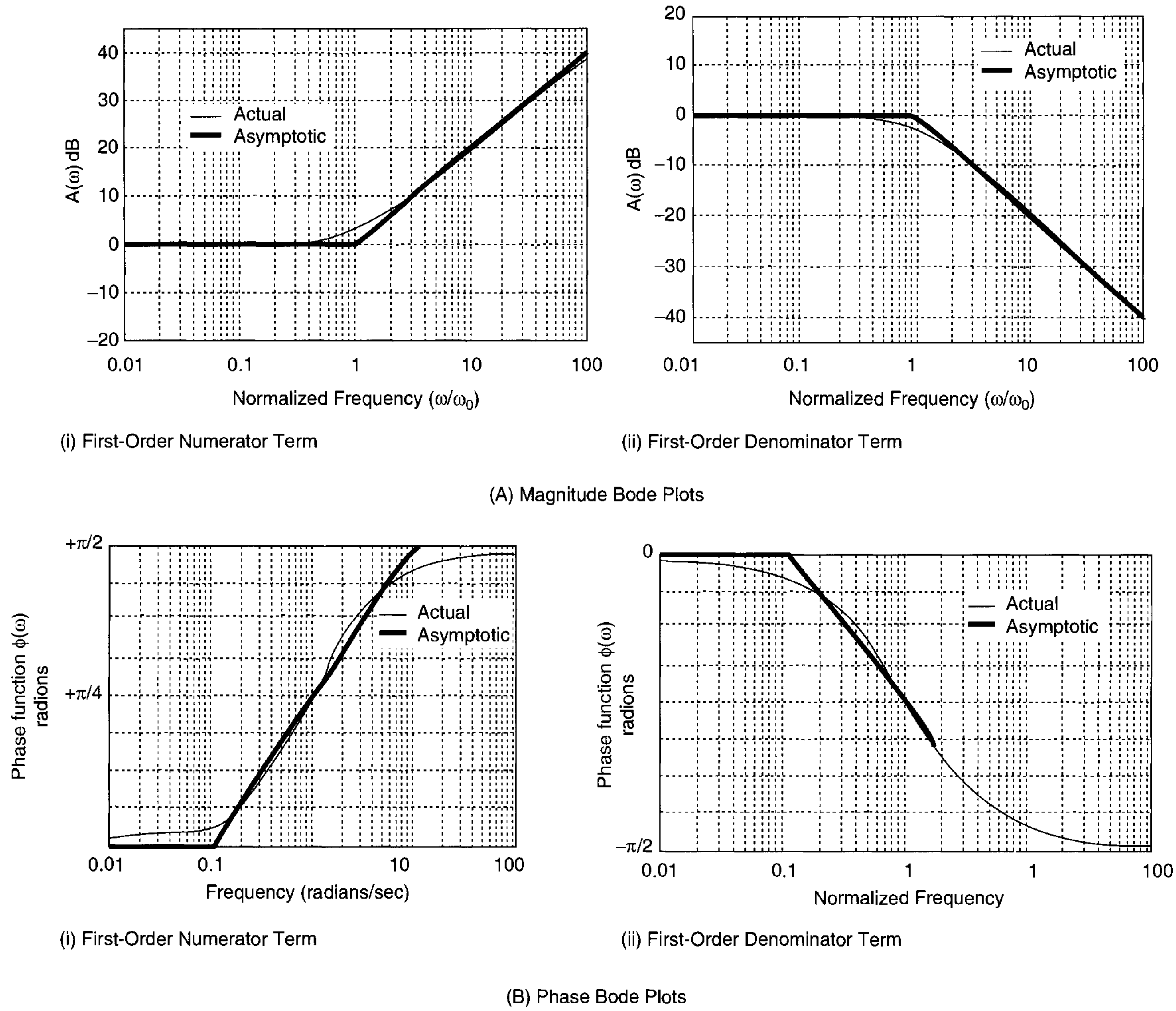


FIGURE 1.31 Bode Diagrams of First-Order Terms

and I are sinusoidal functions of the same frequency. Power in ac circuits is a function of time. For $v(t) = V_m \cos(\omega t + \alpha)$ [V] and $i(t) = I_m \cos(\omega t + \beta)$[A],

$$\begin{aligned} p(t) &= [V_m \cos(\omega t + \alpha)][I_m \cos(\omega t + \beta)] \\ &= 0.5 V_m I_m \{\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\} \end{aligned} \quad (1.73)$$

Average power P is defined as the energy transferred per second and can be calculated by integrating $p(t)$ for 1s. Since the voltage and current are periodic signals of the same frequency, the average power is the energy per cycle multiplied by the frequency f. Energy per cycle is calculated by integrating $p(t)$ over one cycle, that is, one period of the voltage or current wave.

$$\begin{aligned} P &= f\left[\int_0^T p(t)dt\right] = \frac{1}{T}\left[\int_0^T p(t)dt\right] \\ &= \frac{1}{T}\int_0^T 0.5 V_m I_m \{\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\}dt. \end{aligned}$$

Evaluating the integral yields average power:

$$\mathrm{P} = 0.5 V_m I_m \cos(\alpha - \beta)[\mathrm{W}]. \quad (1.74)$$

The $p(t)$ is referred to as the **instantaneous power**, and P is the average power or **ac power**. When a sinusoidal voltage of peak value of V_m [V] is applied to a 1Ω resistor, the average power

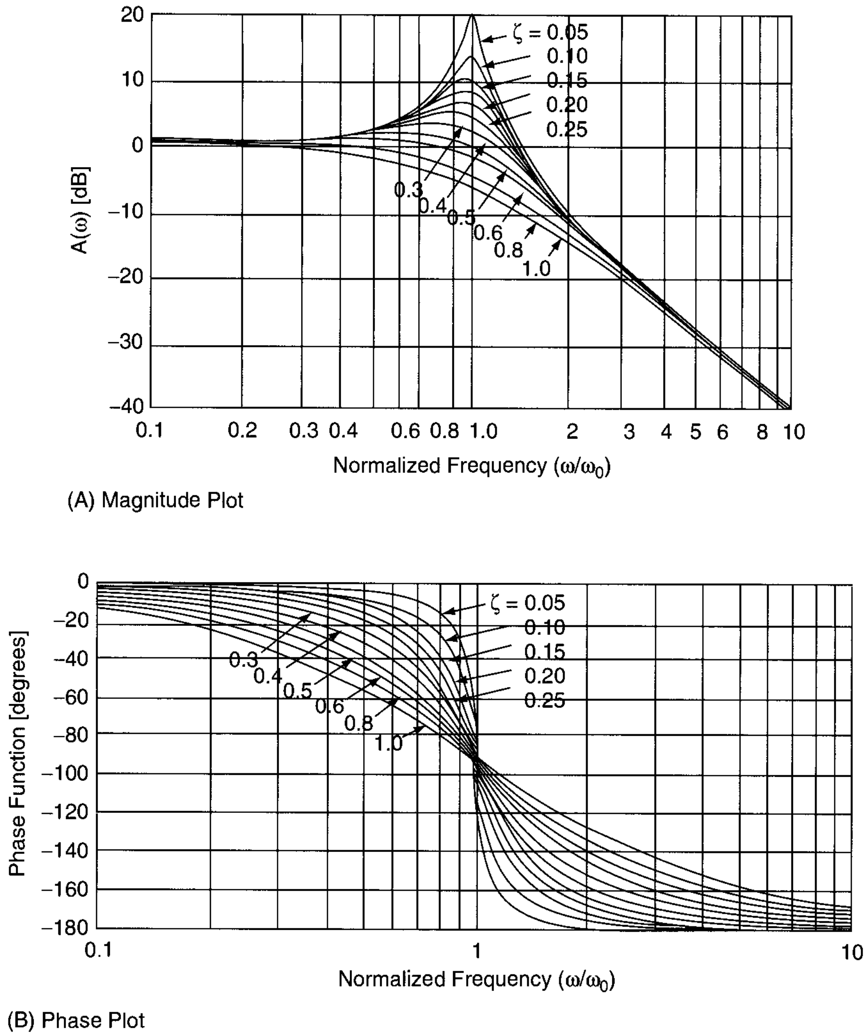


FIGURE 1.32 Bode Diagrams of $H(j\omega) = [(\omega_0^2 - \omega^2) + j(2\zeta\omega_0\omega)]^{-1}$

dissipated in the resistor is 0.5 V_m^2 [W]. Similarly, when a sinusoidal current of I_m [A] passes through a one ohm resistor, the average power dissipated in the resistor is 0.5 I_m^2 [W].

Root mean square (RMS) or effective value of an ac voltage or current is defined as the equivalent dc voltage or current that will dissipate the same amount of power in a $1 - \Omega$ resistor.

$$V_{RMS} = \sqrt{0.5}V_m = 0.707V_m. \qquad (1.75)$$
$$I_{RMS} = \sqrt{0.5}V_m = 0.707I_m.$$

For determining whether the power is dissipated (consumed) or supplied (delivered), source or load power conventions are used. In a load, the current flowing into the positive terminal of the voltage is taken in calculating the dissipated power. For a source, the current leaving the positive terminal of the voltage is used for evaluating the supplied power.

1.10.3 Power Calculations in AC Circuits

The average power of a resistance R in an ac circuit is obtained as $0.5\ V_m^2/R$ [W] or $0.5\,R\,I_m^2$ [W], with V_m the peak voltage across the resistance and I_m the peak current in the resistance. The average power in an inductor or a capacitor in an ac circuit is zero.

Phasor voltages and currents are used in ac calculations. Average power in ac circuits can be expressed in terms of phasor voltage and current. Using the effective values of V and I for the phasors, the following definitions are given:

Using the notation $\boldsymbol{V} = V\angle\alpha$ [V], $\boldsymbol{I} = I\angle\beta$ [A] and conjugate $\boldsymbol{I}$, denoted by $\boldsymbol{I}^* = I\angle - \beta$ [A], you get:

TABLE 1.10 AC Power in Circuit Elements

Element	Voltage–current relationship	Complex power: $S = VI^*$	Average power: P	Reactive power: Q	Power factor
Resistance R [ohm]	$\boldsymbol{V} = R\boldsymbol{I}$	$V\,I/0^0$	I^2R	0	Unity
Inductive reactance $X_L = \omega L$ [ohm]	$\boldsymbol{V} = jX_L\boldsymbol{I}$	$V\,I/90^0$	0	I^2X_L	Zero-lagging
Capacitive susceptance $B_C = \omega C$ [siemens]	$\boldsymbol{I} = jB_C\boldsymbol{V}$	$V\,I/-90^0$	0	$-V^2B_C$	Zero-leading
Impedance $\boldsymbol{Z} = (R + jX) = Z/\theta$ [ohm]	$\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{I}$	$V\,I/\theta = I^2Z$	I^2R	I^2X	$\cos\theta$ $\theta > 0$ Lagging $\theta < 0$ Leading
Admittance $\boldsymbol{Y} = (G + jB) = Y/\phi$ [siemens]	$\boldsymbol{I} = \boldsymbol{Y}\boldsymbol{V}$	$V\,I/-\phi = V^2\boldsymbol{Y}$	V^2G	V^2B	$\cos\phi$ $\phi > 0$ Leading $\phi < 0$ Lagging

Notes: Bold letters refer to phasors or complex numbers. Units: Voltage in volts, current in amperes, average power in watts, reactive power in volt ampere reactive, and complex power in voltampere

$$\text{Average power } P = V\,I\cos(\alpha - \beta) = \text{Real}\{\boldsymbol{V}\,\boldsymbol{I}^*\}[\text{W}] \quad (1.76a)$$

$$\text{Reactive power } Q = V\,I\sin(\alpha - \beta) = \text{Imaginary}\{\boldsymbol{V}\,\boldsymbol{I}^*\}[\text{VAR}] \quad (1.76b)$$

$$\text{Apparent power } S = V\,I[\text{VA}] \quad (1.76c)$$

$$\text{Complex power } S = (P + jQ) = \{\boldsymbol{V}\,\boldsymbol{I}^*\}[\text{VA}] \quad (1.76d)$$

In the set of equations [W] stands for watts, [VAR] for volt-ampere reactive, and [VA] for voltampere. Average power is also called **active power, real power** or simply **power**. Reactive power is also referred to as **imaginary power**. Reactive power is a useful concept in power systems because the system voltage is affected by the reactive power flow. The average power in an inductor or capacitor is zero. By definition, the reactive power taken by an inductor is positive, and the reactive power taken by a capacitor is negative. Complex power representation is useful in calculating the power supplied by the source to a number of loads connected in the system.

Power factor is defined as the ratio of the average power in an ac circuit to the apparent power, which is the product of the voltage and current magnitudes.

$$\text{power factor} = \frac{\text{average power}}{\text{apparent power}} = \frac{P}{S} \quad (1.77)$$

Power factor (PF) has a value between zero and *unity*. The nature of the power factor depends on the relationship between the current and voltage phase angles as:

$(\alpha - \beta) > 0$ PF is lagging.

$(\alpha - \beta) = 0$ PF is unity (UPF).

$(\alpha - \beta) < 0$ PF is leading.

$(\alpha - \beta) = \pi/2$ PF is zero-lagging.

$(\alpha - \beta) = -\pi/2$ PF is zero-leading.

As reactive power $Q = V\,I\,\sin(\alpha - \beta)$,

its sign depends on the nature of PF:
Q is positive for lagging PF.
Q is negative for leading PF.
Q is zero for UPF.

The PF of an inductive load is lagging and that of a capacitive load is leading. A pure inductor has zero-lagging power factor and absorbs positive reactive power. A pure capacitor has zero-leading power factor and absorbs negative reactive power or delivers positive reactive power. Table 1.10 summarizes the expressions for various power quantities in ac circuit elements. Loads are usually specified in terms of P and PF at a rated voltage. Examples are motors and household appliances. Alternatively, the apparent P and PF can be specified.

PF plays an important role in power systems. The product of V and I is called the apparent power. The investment cost of a utility depends on the voltage level and the current carried by the conductors. Higher current needs larger, more expensive conductors. Higher voltage means more insulation costs. The revenue of a utility is generally based on the amount of energy in Kilowatt per hour sold. At low power factors, the revenue is low since power P and energy sold is less. For getting full benefit of investment, the utility would like to sell the highest possible energy, that is, operate at unity power factor all the time. Utilities can have a tariff structure that penalizes the customer for low power factor. Most electromagnetic equipment such as motors have low lagging PF and absorb large reactive powers. By connecting capacitors across the terminals of the motor, part or all of the reactive power absorbed by the motor can be supplied by the capacitors. The reactive power from the supply will be reduced, and the supply PF improved. Cost of the capacitors is balanced against the savings accrued due to PF improvement.