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### 3.1 Introduction

Hand analysis of large circuits, especially if they contain nonlinear elements, can be tedious and prone to error. In such cases, computer-aided design (CAD) is necessary to speed up the analysis procedure and to obtain accurate results. This chapter describes the algorithms used in CAD programs. Such programs are available from commercial companies, such as *Mentor Graphics*, *Cadence*, and *Avant!*

Typically, CAD programs are based on formulating the equations using matrix methods, and the equations are then solved using numerical methods. For ease in programming, methods that are general and straightforward are preferred. This chapter assumes that the reader is familiar with methods for solving a set of simultaneous linear equations.

### 3.2 Modified Nodal Analysis

The first step in circuit analysis is formulation of equations. This section modifies the nodal analysis method described in Chapter 1 of this book's Section 1 to include different types of elements in the formulation. Recall that in nodal analysis, Kirchhoff's current law (KCL) is used to write equations at each node in terms of the nodal voltages and element values. The equations are written for one node at a time, and the node voltages are unknown variables. In CAD, it is useful to consider

one element at a time and to develop the matrix equations on this basis. Assume that the nodes in a circuit are labeled by consecutive integers from 1 to  $N$  such that there are the  $N$  nodal voltages in the circuit. The reference node is usually labeled zero. This presentation uses the lowercase letters  $j$ ,  $k$ ,  $l$ , and  $m$  to denote nodes in the circuit. Voltages will be denoted by  $V$  and branch currents by  $I$ . Consider a linear resistor  $R$  connected between nodes  $j$  and  $k$  as in Figure 3.1.

This element will affect the KCL equations at nodes  $j$  and  $k$ , and the contributions will be:

$$\text{KCL at Node } j: \dots + GV_j - GV_k \dots = 0.$$

$$\text{KCL at Node } k: \dots - GV_j + GV_k \dots = 0.$$

The  $V_j$  and  $V_k$  are the nodal voltages at the two nodes  $V = V_j - V_k$  and  $G = 1/R$ , and the ellipses (...) denote contributions from other elements connected to that node. Thus, the entries in the nodal admittance matrix due to the resistor are in four locations:

	Col.	Col.
	$V_j$	$V_k$
Row $j$ :	$G$	$-G$
Row $k$ :	$-G$	$G$

Row and Col. denote the row and the column in the matrix, respectively. If one end of the resistor, for example node  $k$ , is connected to the reference or ground node, then the

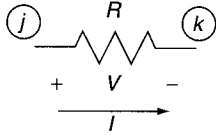


FIGURE 3.1 Linear Resistor

corresponding column and row are not present in the admittance matrix, and there is only one entry in the matrix at location  $(j, j)$ . The above is called the **circuit stamp** for the element and can be used for all elements that can be written in admittance form (i.e., the current through the element can be written in terms of nodal voltages and the element value).

Next, consider an independent voltage source for which the current through it cannot be written in terms of nodal voltages, such as in Figure 3.2.

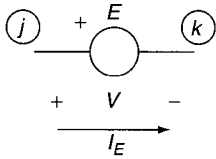


FIGURE 3.2 Independent Voltage Source

In this case, the KCL equations at nodes  $j$  and  $k$  are listed as:

$$\begin{aligned} \text{KCL at node } j: \dots + I_E \dots &= 0. \\ \text{KCL at node } k: \dots - I_E \dots &= 0. \end{aligned} \quad (3.1)$$

The current  $I_E$  is included in the formulation. This increases the number of unknown variables by one. Hence, another equation is needed so that the number of unknowns and the number of equations are the same and can be solved. The additional equation for the voltage source is the following:

$$V_j - V_k = E. \quad (3.2)$$

The circuit stamp for this element includes both equations 3.1 and 3.2 and becomes:

	Col. $V_j$	Col. $V_k$	Col. $I_E$	RHS
Row $j$ :			1	$\begin{bmatrix} \\ \\ E \end{bmatrix}$
Row $k$ :			-1	
Row $I_E$ :	1	-1		

Row  $I_E$  denotes the additional equation 3.2 added to nodal analysis, and Col.  $I_E$  is the additional column in matrix due to the current through the voltage source. The vector RHS denotes the right-hand side and contains known values of sources. This method is called **modified nodal analysis (MNA)** because of the modifications made to nodal analysis to allow different

types of elements in the formulation. The circuit stamps for other common elements are given in Table 3.1 and can be found in CAD books. In this table, rows labeled with lowercase letters  $j, k, l$ , and  $m$  represent KCL equations at that node, and  $V_j, V_k, V_l$ , and  $V_m$  represent nodal voltages at the corresponding nodes. All rows and columns labeled by a current  $I$  represent additional rows and columns required for MNA.

As examples, the MNA equations for the linear circuit are in Figure 3.3.

Scanning of the circuit shows that it has three nodal voltages and that it needs three branch currents for MNA formulation. The branch currents are required, one each, for the voltage source, the inductor in impedance form, and the VVT. The total number of variables is six and can be computed before the matrices are formulated. The MNA equations are as follows:

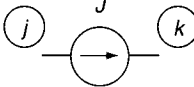
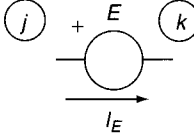
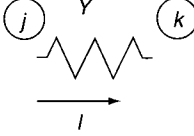
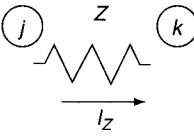
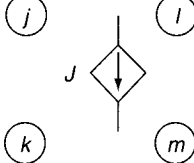
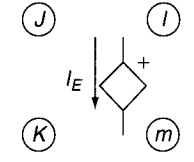
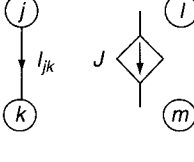
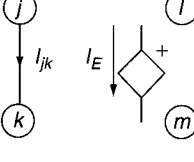
$$\begin{aligned} & \begin{bmatrix} G & -G & 0 & 1 & 0 & 0 \\ -G & G & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\mu & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ I_E \\ I_L \\ I_{VVT} \end{bmatrix} \\ & + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & -C & 0 & 0 & 0 \\ 0 & -C & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ I_E \\ I_L \\ I_{VVT} \end{bmatrix} \\ & = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} E, \quad \mathbf{v}(0) = \begin{bmatrix} V_1(0) \\ V_2(0) \\ V_3(0) \\ I_E(0) \\ I_L(0) \\ I_{VVT}(0) \end{bmatrix} \end{aligned}$$

The first three rows represent KCL equations at the three nodes. The last three rows represent the characteristic equations from the independent voltage source, inductor in impedance form, and VVT, respectively. The equations for linear time-invariant circuits are in the form:

$$\mathbf{G}\mathbf{v}(t) + \mathbf{C} \frac{d}{dt} \mathbf{v}(t) = \mathbf{d}u(t) \text{ and } \mathbf{v}(0) = \mathbf{v}_0, \quad (3.3)$$

where all conductances and constants arising in the formulation are stored in the matrix  $\mathbf{G}$  and all capacitor and inductor values are stored in the matrix  $\mathbf{C}$ . In addition, the connection of the input source to the circuit is in  $\mathbf{d}$ , the vector of nodal voltages and some branch currents needed for MNA is  $\mathbf{v}(t)$ , and the initial condition vector is  $\mathbf{v}_0$ .

TABLE 3.1 Circuit Stamps for MNA

Element	Symbol	Matrix entries
Current source $J$		$\begin{matrix} & V_j & V_k & \text{RHS} \\ \text{Row } j: & \left[ \begin{array}{cc} & \end{array} \right] & & \left[ \begin{array}{c} -J \\ J \end{array} \right] \\ \text{Row } k: & & & \end{matrix}$
Voltage source $E$		$\begin{matrix} & V_j & V_k & I_E & \text{RHS} \\ \text{Row } j: & \left[ \begin{array}{ccc} & & 1 \end{array} \right] & & \left[ \begin{array}{c} \\ \\ E \end{array} \right] \\ \text{Row } k: & & & -1 & \\ \text{Row } I_E: & \left[ \begin{array}{cc} 1 & -1 \end{array} \right] & & & \end{matrix}$
Admittance $I = Y(V_j - V_k)$		$\begin{matrix} & V_j & V_k \\ \text{Row } j: & \left[ \begin{array}{cc} Y & -Y \end{array} \right] \\ \text{Row } k: & \left[ \begin{array}{cc} -Y & Y \end{array} \right] \end{matrix}$
Impedance $V_j - V_k = ZI_Z$		$\begin{matrix} & V_j & V_k & I_Z \\ \text{Row } j: & \left[ \begin{array}{ccc} & & 1 \end{array} \right] \\ \text{Row } k: & & & -1 \\ \text{Row } I_Z: & \left[ \begin{array}{ccc} 1 & -1 & -Z \end{array} \right] \end{matrix}$
Voltage-controlled current source (VCT) $J = g(V_j - V_k)$		$\begin{matrix} & V_j & V_k & V_l & V_m \\ \text{Row } j: & \left[ \begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array} \right] \\ \text{Row } k: & & & & \\ \text{Row } l: & \left[ \begin{array}{cc} g & -g \end{array} \right] & & & \\ \text{Row } m: & \left[ \begin{array}{cc} -g & g \end{array} \right] & & & \end{matrix}$
Voltage-controlled voltage source (VVT) $V_l - V_m = \mu(V_j - V_k)$		$\begin{matrix} & V_j & V_k & V_l & V_m & I_E \\ \text{Row } j: & \left[ \begin{array}{cccc} & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right] \\ \text{Row } k: & & & & & \\ \text{Row } l: & & & & & 1 \\ \text{Row } m: & & & & & -1 \\ \text{Row } I_E: & \left[ \begin{array}{cccc} -\mu & \mu & 1 & -1 \end{array} \right] & & & \end{matrix}$
Current-controlled current source (CCT) $J = \beta I_{jk}$		$\begin{matrix} & V_j & V_k & V_l & V_m & I_{jk} \\ \text{Row } j: & \left[ \begin{array}{cccc} & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right] \\ \text{Row } k: & & & & & -1 \\ \text{Row } l: & & & & & \beta \\ \text{Row } m: & & & & & -\beta \\ \text{Row } I_{jk}: & \left[ \begin{array}{cc} 1 & -1 \end{array} \right] & & & \end{matrix}$
Current-controlled voltage source (CVT) $V_l - V_m = rI_{jk}$		$\begin{matrix} & V_j & V_k & V_l & V_m & I_{jk} & I_E \\ \text{Row } j: & \left[ \begin{array}{cccc} & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right] \\ \text{Row } k: & & & & & -1 & \\ \text{Row } l: & & & & & & 1 \\ \text{Row } m: & & & & & & -1 \\ \text{Row } I_{jk}: & \left[ \begin{array}{cc} 1 & -1 \end{array} \right] & & & & & \\ \text{Row } I_E: & & & \left[ \begin{array}{cc} 1 & -1 \end{array} \right] & & -r & \end{matrix}$

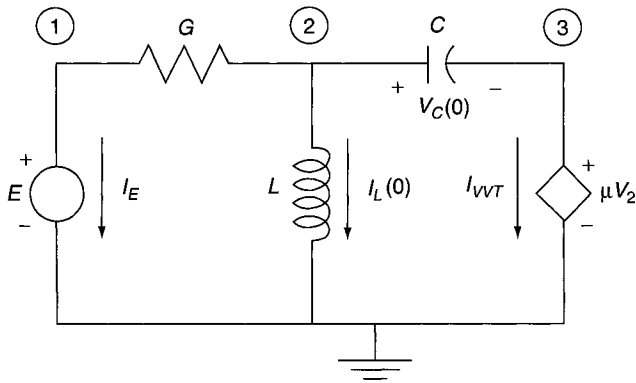


FIGURE 3.3 Linear Circuit Example

For sinusoidal steady state response of a linear time invariant circuit, the input is:

$$u(t) = e^{j\omega t}$$

Here, the initial conditions  $v_0$  are ignored, and phasor analysis is used for computing the results. In this case, equation 3.3 becomes:

$$(G + j\omega C)V(j\omega) = d, \quad (3.4)$$

where  $V(j\omega)$  is the phasor representation of  $v(t)$ . The matrix equation 3.4 is a set of simultaneous linear equations and are solved numerically for  $V(j\omega)$ . The solution, in general, is a complex quantity with magnitude  $|V(j\omega)|$  and phase angle  $\angle V(j\omega)$ :

$$V(j\omega) = |V(j\omega)|\angle V(j\omega).$$

The magnitude is also plotted in decibels [dB] and computed as:

$$|V(j\omega)|_{dB} = 20 \log |V(j\omega)|.$$

Using the values  $L = 1$  H,  $G = 1$  S,  $C = 1$  F,  $\mu = 0.5$ , the magnitude of  $V_2$  is plotted in Figure 3.4.

### 3.3 Formulation of MNA Equations of Nonlinear Circuits

Formulation of MNA equations of nonlinear circuits follows the same steps as equation formulation for linear circuits. Consider the circuit in Figure 3.5 with a nonlinear resistor defined by  $I_R = g(V_R) = 0.001(V_R)^3$  and a nonlinear capacitor defined by  $Q_C = q(V_C) = 0.001(V_C)^3$ . The current through the capacitor is given by  $I_C = (d/dt)Q_C$ . Both the nonlinear elements are called voltage-controlled elements because the current through the resistor and charge on the capacitor are nonlinear functions of the voltage across the individual elements. Let the input be a DC source  $E(t) = 1$  V.

The MNA equations for the circuit in Figure 3.5 are the following:

$$\begin{aligned} \frac{V_1 - V_2}{1000} + I_E = 0 = f_1, \\ \frac{V_2 - V_1}{1000} + \frac{(V_2)^3}{1000} + \frac{d}{dt} Q_C(V_2) = 0 = f_2, \\ V_1 - E = 0 = f_3. \end{aligned} \quad (3.5)$$

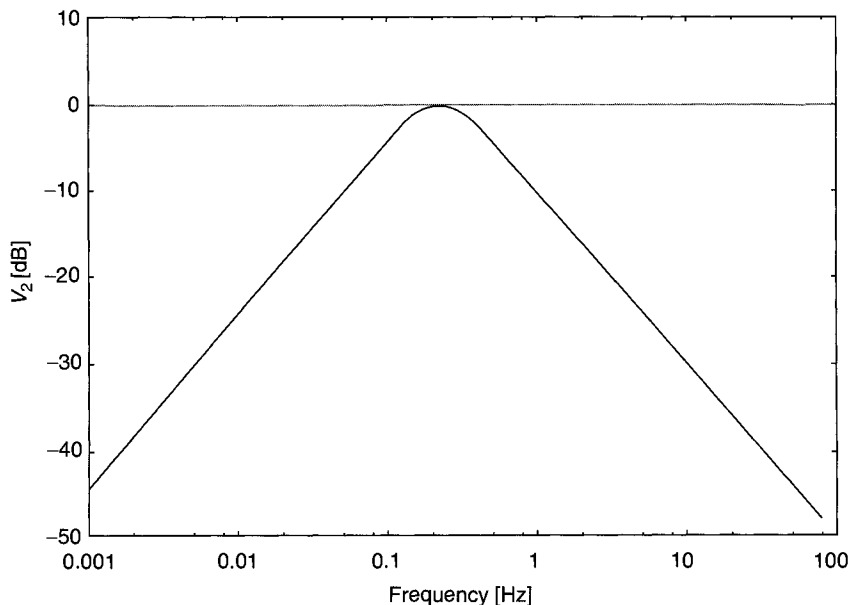


FIGURE 3.4 Frequency Response of a Linear Circuit

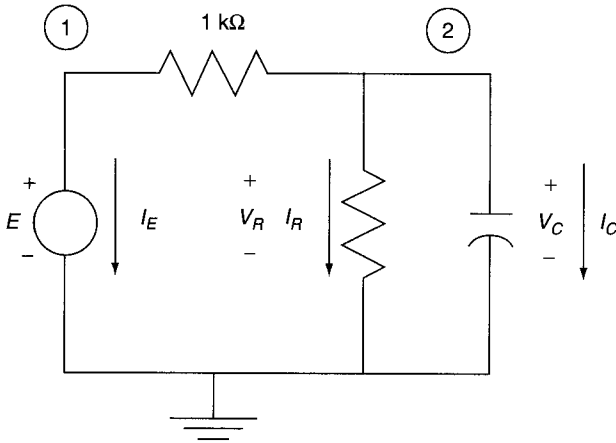


FIGURE 3.5 Nonlinear Circuit

The first two equations are the KCL equations at nodes 1 and 2 respectively, and the last equation is the characteristic equation for the voltage source. In general, all the equations are nonlinear and are collectively written in the form

$$f(\dot{\mathbf{v}}, \mathbf{v}, t) = \begin{bmatrix} f_1(\dot{\mathbf{v}}, \mathbf{v}, t) \\ f_2(\dot{\mathbf{v}}, \mathbf{v}, t) \\ f_3(\dot{\mathbf{v}}, \mathbf{v}, t) \end{bmatrix} = \mathbf{0}, \quad (3.6)$$

where the unknown vector and initial conditions are the following:

$$\mathbf{v} = \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix} \text{ and } \mathbf{v}(0) = \mathbf{v}_0.$$

### 3.4 A Direct Current Solution of Nonlinear Circuits

For a direct current (dc) solution of nonlinear circuits, the derivatives of the network variables and time are set to zero, and equation 3.6 reduces to:

$$f(\mathbf{0}, \mathbf{v}, 0) = \mathbf{x}(\mathbf{v}) = \mathbf{0}.$$

$$\mathbf{x}(\mathbf{v}^{i+1}) \approx \begin{bmatrix} x_1(\mathbf{v}^i) + \frac{\partial x_1}{\partial V_1} (V_1^{i+1} - V_1^i) + \frac{\partial x_1}{\partial V_2} (V_2^{i+1} - V_2^i) + \frac{\partial x_1}{\partial I_E} (I_E^{i+1} - I_E^i) \\ x_2(\mathbf{v}^i) + \frac{\partial x_2}{\partial V_1} (V_1^{i+1} - V_1^i) + \frac{\partial x_2}{\partial V_2} (V_2^{i+1} - V_2^i) + \frac{\partial x_2}{\partial I_E} (I_E^{i+1} - I_E^i) \\ x_3(\mathbf{v}^i) + \frac{\partial x_3}{\partial V_1} (V_1^{i+1} - V_1^i) + \frac{\partial x_3}{\partial V_2} (V_2^{i+1} - V_2^i) + \frac{\partial x_3}{\partial I_E} (I_E^{i+1} - I_E^i) \end{bmatrix} \approx \mathbf{0} \quad (3.8)$$

For example, the equations for the circuit in Figure 3.5 are the following:

$$\mathbf{x}(\mathbf{v}) = \begin{bmatrix} x_1(V_1, V_2, I_E) \\ x_2(V_1, V_2, I_E) \\ x_3(V_1, V_2, I_E) \end{bmatrix} = \begin{bmatrix} \frac{V_1 - V_2}{1000} + I_E \\ \frac{V_2 - V_1}{1000} + 0.001(V_2)^3 \\ V_1 - E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \quad (3.7)$$

The nonlinear equation 3.7 is solved numerically using an iterative method called the Newton–Raphson (NR) method. Let  $\mathbf{v}^0$  denote the initial guess and  $\mathbf{v}^i$  the result of the  $i$ th iteration for the solution of equation 3.7. The calculation of the next iteration value  $\mathbf{v}^{i+1}$  is attempted such that  $\mathbf{x}(\mathbf{v}^{i+1}) \approx \mathbf{0}$ . Expanding  $\mathbf{x}(\mathbf{v}^{i+1})$  in a Taylor series around the point  $\mathbf{v}^i$  gives Equation 3.8.

Equation 3.8 shows that only the first two terms in the series have been used and other higher order terms have been ignored. Equation 3.8 can be rewritten in the form:

$$\mathbf{J}^i \Delta \mathbf{v}^i = -\mathbf{x}(\mathbf{v}^i), \quad (3.9)$$

where

$$\mathbf{J}^i = \begin{bmatrix} \frac{\partial x_1}{\partial V_1} & \frac{\partial x_1}{\partial V_2} & \frac{\partial x_1}{\partial I_E} \\ \frac{\partial x_2}{\partial V_1} & \frac{\partial x_2}{\partial V_2} & \frac{\partial x_2}{\partial I_E} \\ \frac{\partial x_3}{\partial V_1} & \frac{\partial x_3}{\partial V_2} & \frac{\partial x_3}{\partial I_E} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1000 & 1000 & 0 \\ -1 & 1 + 3(V_2^i)^2 & 0 \\ 1000 & 1000 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Delta \mathbf{v}^i = \begin{bmatrix} V_1^{i+1} - V_1^i \\ V_2^{i+1} - V_2^i \\ I_E^{i+1} - I_E^i \end{bmatrix},$$

and  $\mathbf{J}^i$  is called the Jacobian of the system, and  $\Delta \mathbf{v}^i$  is the change in the solution at the  $i$ th iteration. From the solution of equation 3.9,  $\Delta \mathbf{v}^i$  is obtained and  $\mathbf{v}^{i+1} = \mathbf{v}^i + \Delta \mathbf{v}^i$  is computed. Note that equation 3.9 is a linear system of equations, and the Jacobian can be created using circuit stamps, as done

previously for linear circuits. The MNA circuit stamp for a nonlinear voltage-controlled resistor  $I_R = g(V_R)$ , shown in Figure 3.1, is as follows:

$$\begin{array}{ccc} & \text{Col.} & \text{Col.} & \\ & V_j & V_k & \\ & & & \text{RHS} \\ \text{Row } j: & \left[ \begin{array}{cc} \frac{\partial g}{\partial V_j} & -\frac{\partial g}{\partial V_k} \end{array} \right] & & \left[ \begin{array}{c} -g(V_R^i) \end{array} \right] \\ \text{Row } k: & \left[ \begin{array}{cc} -\frac{\partial g}{\partial V_j} & \frac{\partial g}{\partial V_k} \end{array} \right] & & \left[ \begin{array}{c} g(V_R^i) \end{array} \right] \end{array}$$

The  $V_R = V_j - V_k$  and the linear resistor are special cases of the nonlinear resistor. The difference from MNA formulation of linear circuits is that, for nonlinear circuits, there exists an entry in RHS for all elements. For a current-controlled nonlinear resistor in the form  $V_R = r(I_R)$ , the impedance form of MNA formulation is used, and the current through the resistor  $I_R$  is added to the vector of the unknowns. The additional equation is  $V_j - V_k - r(I_R) = 0$ .

The Newton–Raphson method is summarized in the following steps:

1. Set the iteration count  $i = 0$ , and estimate the initial guess of  $\mathbf{v}^0$ .
2. Calculate the Jacobian  $\mathbf{J}^i$  and right-hand side of equation 3.9, which is  $-\mathbf{x}(\mathbf{v}^i)$ .
3. Solve equation 3.9 for  $\Delta\mathbf{v}^i$ .
4. Update the solution vector  $\mathbf{v}^{i+1} = \mathbf{v}^i + \Delta\mathbf{v}^i$ .
5. Calculate the error in the solution as the norm of the vector  $\|\mathbf{x}(\mathbf{v}^{i+1})\|$ . A common method is to use the norm defined as  $L_2 = \|\mathbf{x}(\mathbf{v}^{i+1})\|_2 = \sqrt{\sum_j (x_j)^2}$ . If the error is small then stop, or instead set  $i = i + 1$  and go to step 2.

Under fairly general conditions, it can be shown that if the initial guess is close to the solution, then the Newton–Raphson method converges quadratically to the solution. For the circuit in Figure 3.6, if the initial guess  $\mathbf{v}^0 = [0 \ 0 \ 0]^T$  is used, then the iterations for nodal voltage  $V_2$  are given in Table 3.2.

Note that the last column in the table shows that the right-hand side of equation 3.9 approaches zero with each iteration, indicating that the final value is being reached and the iterations can be stopped when the value is sufficiently close to zero. Also note that with each iteration, the change in the solution is becoming smaller, indicating that one is getting closer to the exact solution. Since the circuit is nonlinear, the Jacobian must be computed for each iteration and is usually the most expensive portion of the entire solution method. If a better initial guess is made, such as  $\mathbf{v}^0 = [1.0 \ 0.7 \ -0.003]^T$ , then the NR method will converge to the final value in fewer iterations.

The algorithm just described is the basic Newton–Raphson method that works well in most, but not all, cases. Modifications to this method as well as to other methods for solving nonlinear equations can be found in references listed at the end of this chapter.

TABLE 3.2 Newton–Raphson Iterations

Iteration	$V_2^{i+1}$	$\Delta V_2^i$	$\ \mathbf{x}(\mathbf{v}^{i+1})\ _2$
0	1.0000	1.0000	0.0316
1	0.7500	-0.2500	0.0131
2	0.6860	-0.0640	0.0030
3	0.6823	-0.0037	0.0002
4	0.6823	-0.0000	0.0000

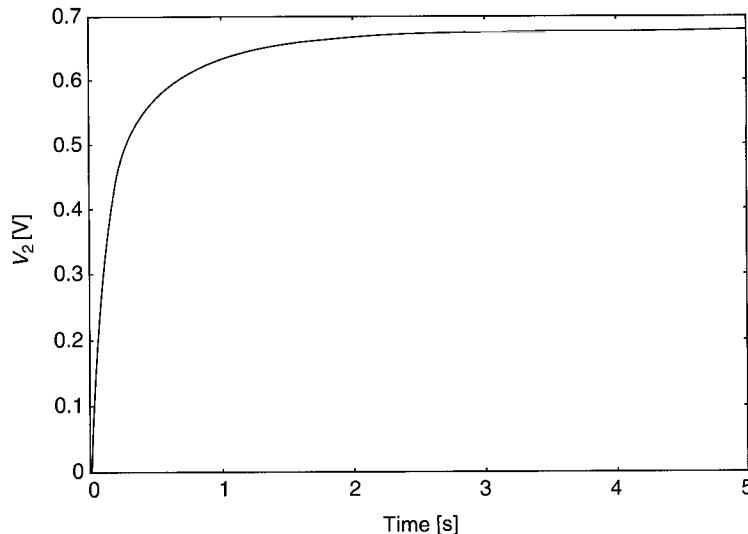


FIGURE 3.6 Time Response of a Nonlinear Circuit

### 3.5 Transient Analysis of Nonlinear Circuits

Another very useful analysis is the time response of nonlinear circuits. This requires the solution of the differential equation 3.6, and once again numerical methods are used. Usually the solution of the network at time  $t = 0$  is known, and the response is calculated after a time step of  $h$  at  $t + h$ . All numerical solution methods replace the time derivative in equation 3.6 with a suitable approximation, and then the resulting equations are solved. Expanding the solution in a Taylor series, the following is obtained:

$$\mathbf{v}(t + h) = \mathbf{v}(t) + h \frac{d}{dt} \mathbf{v}(t) + \dots \text{ higher order terms.}$$

Assuming that the time step  $h$  is small and ignoring the higher order terms as being small, the method called the Euler forward method is used. The method approximates the derivative with:

$$\dot{\mathbf{v}}(t) = \frac{d}{dt} \mathbf{v}(t) \approx \frac{\mathbf{v}(t + h) - \mathbf{v}(t)}{h}. \quad (3.10)$$

Substituting equation 3.10 in equation 3.6 yields:

$$\mathbf{f}\left(\frac{\mathbf{v}(t + h) - \mathbf{v}(t)}{h}, \mathbf{v}(t), t\right) = 0 = \mathbf{x}(\mathbf{v}(t + h)),$$

which is a nonlinear equation with the unknown value  $\mathbf{v}(t + h)$ . It can be written in the form  $\mathbf{x}(\mathbf{v}(t + h)) = 0$  and solved using the Newton–Raphson method given previously. Experience has shown that this method works well with small step sizes; however, large step sizes give results that diverge from the true solution of the network. Such methods are called unstable and are generally not used in integration of differential equations. A better solution method is to use the Euler backward method for which the derivative is approximated by a Taylor series expansion around the time point  $\mathbf{v}(t + h)$ :

$$\mathbf{v}(t) = \mathbf{v}(t + h) + (-h) \frac{d}{dt} \mathbf{v}(t + h) + \dots$$

higher order terms.

This equation can be rewritten in the form:

$$\dot{\mathbf{v}}(t + h) = \frac{d}{dt} \mathbf{v}(t + h) \approx \frac{\mathbf{v}(t + h) - \mathbf{v}(t)}{h}. \quad (3.11)$$

Substituting equation 3.11 in equation 3.6 results in:

$$\mathbf{f}\left(\frac{\mathbf{v}(t + h) - \mathbf{v}(t)}{h}, \mathbf{v}(t + h), t + h\right) = 0 = \mathbf{x}(\mathbf{v}(t + h)), \quad (3.12)$$

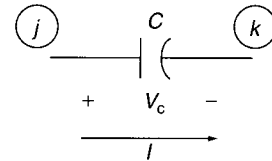
which is a nonlinear equation in the variable  $\mathbf{v}(t + h)$  and is solved using the Newton–Raphson method. The last step is called the discretization of the differential equation, and this step changes it into an algebraic equation. For example, substituting equation 3.11 in equation 3.5 results in:

$$\begin{bmatrix} \frac{V_1(t + h) - V_2(t + h)}{1000} + I_E(t + h) \\ \frac{V_2(t + h) - V_1(t + h)}{1000} + \frac{(V_2(t + h))^3}{1000} + \frac{Q_C(V_2(t + h)) - Q_C(V_2(t))}{h} \\ V_1(t + h) - E(t + h) \end{bmatrix} = \mathbf{0},$$

where the Jacobian will be:

$$\mathbf{J}^i = \begin{bmatrix} \frac{1}{1000} & \frac{-1}{1000} & 1 \\ -\frac{1}{1000} & \frac{1 + 3(V_2^i)^2}{1000} + \frac{3(V_2^i)^2}{1000h} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The explicit dependence on time has been dropped for simplicity. The Jacobian can be created using circuit stamps for the nonlinear elements as done in the case of dc solution of nonlinear circuits. For a nonlinear voltage-controlled capacitor, such as:



defined by the equation  $Q_C = q(V_C) = 0.001(V_C)^3$  where  $V_C = V_j - V_k$ , the circuit stamp is obtained by writing KCL equations at nodes  $j$  and  $k$ :

$$\text{KCL at node } j: \dots + \frac{d}{dt} Q_C(t + h) \dots = 0.$$

$$\text{KCL at node } k: \dots - \frac{d}{dt} Q_C(t + h) \dots = 0.$$

Using the Euler backward method, the differentials are discretized by:

$$\text{KCL at node } j: \dots + \frac{Q_C(t + h) - Q_C(t)}{h} \dots = 0.$$

$$\text{KCL at node } k: \dots - \frac{Q_C(t + h) - Q_C(t)}{h} \dots = 0.$$

The Jacobian and RHS entries become:

$$\begin{array}{l}
 \text{Col.} \\
 V_j \\
 \text{Row } j: \left[ \begin{array}{c} \frac{1}{h} \frac{\partial Q_C}{\partial V_j} \\ \frac{1}{h} \frac{\partial Q_C}{\partial V_j} \end{array} \right] \\
 \text{Row } k: \left[ \begin{array}{c} -\frac{1}{h} \frac{\partial Q_C}{\partial V_j} \\ \frac{1}{h} \frac{\partial Q_C}{\partial V_k} \end{array} \right] \\
 \text{Col.} \\
 V_k \\
 \text{RHS} \\
 \left[ \begin{array}{c} -Q_C(V_C^i(t+h)) - Q_C(V_C(t)) \\ \frac{h}{h} \\ Q_C(V_C^i(t+h)) - Q_C(V_C(t)) \\ \frac{h}{h} \end{array} \right]
 \end{array}
 \quad (3.13)$$

Note that in deriving equation 3.13, the use of the chain rule of differentiation and then discretizing  $(d/dt)V_C$  has been avoided:

$$\frac{d}{dt}Q_C = \frac{d}{dV_C}Q_C \frac{d}{dt}V_C.$$

The value of  $(d/dt)V_C$  has also been discretized. Discretization applied directly to the charge of the capacitor (and flux in case of inductors) leads to better results that maintain charge (and flux) conservation in the numerical solution of electrical circuits. Additional details can be found in work by Kundert (1995).

For the circuit in Figure 3.5, let the capacitor be initially discharged such that  $V_2(0) = 0$  and the correct initial condition vector is  $\nu(0) = [1.0 \ 0.0 \ -0.001]^T$ . For this circuit, it is expected that the capacitor will charge up from 0 V to a finite value with time. All nodal voltages in the circuit will remain less than the source voltage of 1 V. If step  $h = 0.1$  s is taken, and the initial guess is  $\nu^0(h) = \nu(0)$ , the Newton–Raphson iteration results will be obtained as shown in Table 3.3.

Once the solution at  $t = 0.1$  s is obtained, the values from the circuit conditions at that time are used, another time step is taken, and the response at  $t = 2h = 0.2$  s is calculated. This procedure is repeated until the end time of interest is reached for computing the response. Although the circuit has a continuous response, the numerical methods are used to compute the response at discrete instants of time. It is expected that if small time steps are taken, then no interesting features of the response will be missed. Note that at each time step, the Newton–Raphson method is used to solve a set of nonlinear equations. Moreover, since the circuit is nonlinear, the Jacobian has to be recomputed at each iteration and each time step. The complete time domain response of the circuit is given in

**TABLE 3.3** Newton–Raphson Iterations for  $t = 0$  and  $h = 0.1$  s

Iteration $i$	$V_2^{i+1}$	$\Delta V_2^i$
0	1.0000	1.0000
1	0.6765	-0.3235
2	0.4851	-0.1914
3	0.4006	-0.0845
4	0.3835	-0.0171
5	0.3828	-0.0007

Figure 3.6. The final value  $V_2 = 0.6823$  V is reached at about  $t = 8$  s. Note that the final value is the same as that obtained in the dc solution above.

Choosing a large step size  $h = 10$  s and calculating the time response will give the results  $V_2(10) = 0.6697$  V,  $V_2(20) = 0.6816$  V,  $V_2(30) = 0.6823$  V, ... Note that even though the result at  $t = 10$  s is not accurate, the error does not grow with each time step and the correct value is reached after a few more steps. Such time domain integration methods are called stable and are preferred over the Euler Forward method that may not give correct results for large time steps.

Finally, this section discusses some issues concerning the use of numerical methods for the solution of differential equations and gives some recommendations for obtaining correct results:

1. The discretization equation 3.11 is an approximation, since only the first two terms in the Taylor series expansion are included. As a result of this approximation, the numerically computed results contain an error at each time step, called the local truncation error (LTE). In general, the smaller the time step used, the smaller will be the LTE. The normal expectation is that if the error is reduced at each time step, then the overall error in the numerical results will also be small.
2. This discussion concerning the LTE suggests that if a discretization formula is used that matches more than two terms in the Taylor series expansion, then more accurate results will be obtained. This is also correct, provided the same step size is used for both discretization methods, or alternatively the second method will allow larger step sizes for a given error.
3. This discussion also suggests that it is not necessary to use the same time step when determining the time domain solution of a circuit over a long period of time. It is possible to change the time step during the computation based on error and accuracy requirements.
4. The Newton–Raphson method is used at each time step to solve a set of nonlinear equations. For this method, a good initial guess gives the solution in fewer iterations. Such formulae, called predictor formulae, are available and reduce the overall computation needed for the solution.
5. The issue of stability of the response with large step sizes has already been mentioned. A stable integration method is preferred. Stable methods ensure that the error at each time step does not accumulate over time and reasonably accurate results are obtained.

Additional reasons for these recommendations, as well as other integration methods for time domain solution of nonlinear circuits, can be found in the references at the end of this chapter.



## References

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