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Circuit Analysis: A Graph-Theoretic Foundation

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2.1 Introduction

The theory of graphs has played a fundamental role in discovering structural properties of electrical circuits. This should not be surprising because graphs, as the reader shall soon see, are good pictorial representations of circuits and capture all their structural characteristics. This chapter develops most results that form the foundation of graph theoretic study of electrical circuits. A comprehensive treatment of these developments may be found in Swamy and Thulasiraman (1981). All theorems in this chapter are stated without proofs.

The development of graph theory in this chapter is self-contained except for the definitions of standard and elementary results from set theory and matrix theory.

2.2 Basic Concepts and Results

2.2.1 Graphs

A graph $G = (V, E)$ consists of two sets: a finite set $V = (v_1, v_2, \dots v_n)$ of elements called **vertices** and a finite set $E = (e_1, e_2, \dots e_n)$ of elements called **edges**. If the edges of G are identified with ordered pairs of vertices, then G is called a **directed** or an **oriented graph**; otherwise, it is called an **undirected** or an **unoriented graph**.

Graphs permit easy pictorial representations. In a pictorial representation, each vertex is represented by a dot, and each edge is represented by a line joining the dots associated with the edge. In directed graphs, an orientation or direction is assigned to each edge. If the edge is associated with the ordered pair (v_i, v_j), then this edge is oriented from v_i to v_j. If an edge e connects vertices v_i and v_j, then it is denoted by $e = (v_i, v_j)$. In a directed graph, (v_i, v_j) refers to an edge directed from v_i to v_j. An undirected graph and a directed graph are shown in Figure 2.1. Unless explicitly stated, the term **graph** may refer to a directed graph or an undirected graph.

The vertices v_i and v_j associated with an edge are called the **end vertices** of the edge. All edges having the same pair of end vertices are called **parallel edges**.

In a directed graph, parallel edges refer to edges connecting the same pair of vertices v_i and v_j the same way from v_i to v_j or from v_j to v_i. For instance, in the graph of Figure 2.1(A), the edges connecting v_1 and v_2 are parallel edges. In the directed graph of Figure 2.1(B), the edges connecting v_3 and v_4 are parallel edges. However, the edges connecting v_1 and v_2 are not parallel edges because they are not oriented in the same way.

If the end vertices of an edge are not distinct, then the edge is called a **self-loop**. The graph of Figure 2.1(A) has one self loop, and the graph of Figure 2.1(B) has two self-loops. An edge is said to be **incident on** its end vertices. In a directed

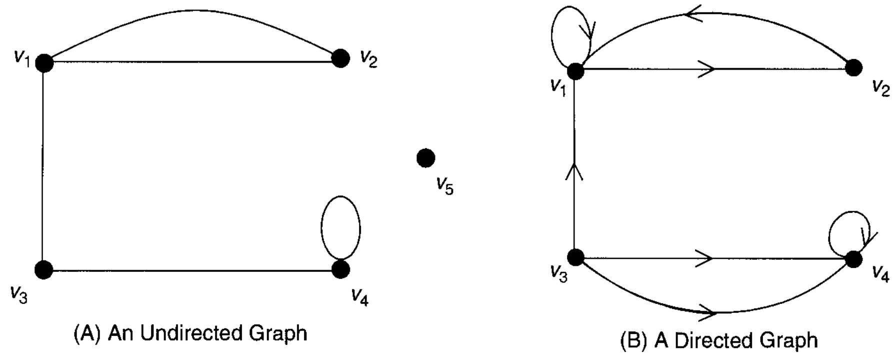


FIGURE 2.1 Graphs: Easy Pictorial Representations

graph, the edge (v_i, v_j) is said to be **incident out** of v_i and is said to be **incident into** v_j. Vertices v_i and v_j are adjacent if an edge connects v_i and v_j.

The number of edges incident on a vertex v_i is called the **degree** of v_i and is denoted by $d(v_i)$. In a directed graph, $d_{in}(v_i)$ refers to the number of edges incident into the vertex v_i, and it is called the **in-degree**. In a directed graph, $d_{out}(v_i)$ refers to the number of edges incident out of the vertex v_i or the **out-degree**. If $d(v_i) = 0$, then v_i is said to be an **isolated vertex**. If $d(v_i) = 1$, then v_i is said to be a **pendant vertex**.

A self-loop at a vertex v_i is counted twice while computing $d(v_i)$. As an example in the graph of Figure 2.1(A), $d(v_1) = 3$, $d(v_4) = 3$, and v_5 is an isolated vertex. In the directed graph of Figure 2.1(B), $d_{in}(v_1) = 3$ and $d_{out}(v_1) = 2$.

Note that in a directed graph, for every vertex v_i,

$$d(v_i) = d_{in}(v_i) + d_{out}(v_i).$$

2.2.2 Basic Theorems

Theorem 2.1: (1) The sum of the degrees of the vertices of a graph G is equal to $2\,m$, where m is the number of edges of G, and (2) in a directed graph with m edges, the sum of the in-degrees and the sum of out-degrees are both equal to m.

The following theorem is known to be the first major result in graph theory.

Theorem 2.2: The number of vertices of odd degree in any graph is even.

Consider a graph $G = (V', E')$. The graph $G' = (V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. As an example, a graph G and a subgraph of G are shown in Figure 2.2.

In a graph G, a **path P** connecting vertices v_i and v_j is an alternating sequence of vertices and edges starting at v_i and ending at v_j, with all vertices except v_i and v_j being distinct. In a directed graph, a path P connecting vertices v_i and v_j is called a **directed path** from v_i to v_j if all the edges in P are oriented in the same direction when traversing P from v_i toward v_j.

If a path starts and ends at the same vertex, it is called a **circuit**. In a directed graph, a circuit in which all the edges are oriented in the same direction is called a **directed circuit**. It is often convenient to represent paths and circuits by the sequence of edges representing them.

For example, in the undirected graph of Figure 2.3(A), P: e_1, e_2, e_3, e_4 is a path connecting v_1 and v_5, and C: e_1,

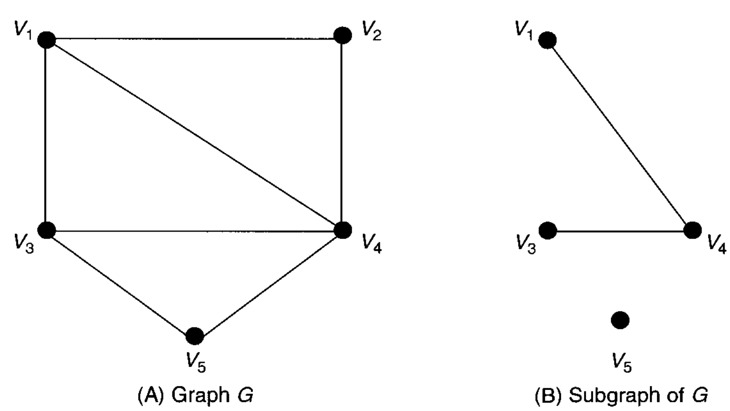


FIGURE 2.2 Graphs and Subgraphs

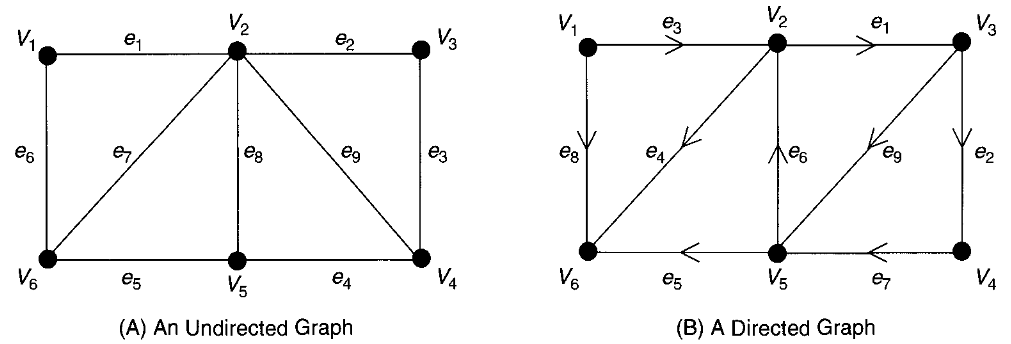


FIGURE 2.3 Graphs Showing Paths

e_2, e_3, e_4, e_5, e_6 is a circuit. In the directed graph of Figure 2.3(B) P: e_1, e_2, e_7, e_5 is a directed path and C: e_1, e_2, e_7, e_6 is a directed circuit. Note that C: e_7, e_5, e_4, e_1, e_2 is a circuit in this directed graph, although it is not a directed circuit. Similarly, P: e_9, e_6, e_3 is a path but not a directed path.

A graph is **connected** if there is a path between every pair of vertices in the graph; otherwise, the graph is not connected. For example, the graph in Figure 2.2(A) is a connected graph, whereas the graph in Figure 2.2(B) is not a connected graph.

A **tree** is a graph that is connected and has no circuits. Consider a connected graph G. A subgraph of G is a **spanning tree** of G if the subgraph is a tree and contains all the vertices of G. A tree and a spanning tree of the graph of Figure 2.4(A) are shown in Figures 2.4(B) and (C), respectively. The edges of a spanning tree T are called the **branches** of T. Given a spanning tree of connected graph G, the **cospanning tree** relative to T is the subgraph of G induced by the edges that are not present in T. For example, the cospanning tree relative to the spanning tree T of Figure 2.4(C) consists of these edges: e_3, e_6, e_7. The edges of a cospanning tree are called **chords**.

It can be easily verified that in a tree, exactly one path connects any two vertices. It should be noted that a tree is minimally connected in the sense that removing any edge from the tree will result in a disconnected graph.

Theorem 2.3: A tree on n vertices has $n-1$ edges.

If a connected graph G has n vertices and m edges, then the **rank** ρ and **nullity** μ of G are defined as follows:

$$\rho(G) = n - 1.$$
$$\mu(G) = m - n + 1.$$

The concepts of rank and nullity have parallels in other branches of mathematics such as matrix theory.

Clearly, if G is connected, then any spanning tree of G has $\rho = n - 1$ branches and

$$\mu = m - n + 1 \text{ chords.}$$

2.2.3 Cuts, Circuits, and Orthogonality

Introduced here are the notions of a cut and a cutset. This section develops certain results that bring out the dual nature of circuits and cutsets.

Consider a connected graph $G = (V, E)$ with n vertices and m edges. Let V_1 and V_2 be two mutually disjoint nonempty subsets of V such that $V = V_1 \cup V_2$. Thus V_2 is the complement of V_1 in V and vice versa. V_1 and V_2 are also said to form a partition of V. Then the set of all those edges that have one end vertex in V_1 and the other in V_2 is called a **cut** of G and is denoted by $< V_1, V_2 >$. As an example, a graph and a cut $< V_1, V_2 >$ of G are shown in Figure 2.5.

The graph G, which results after removing the edges in a cut, will not be connected. A **cutset** S of a connected graph G is a minimal set of edges of G, such that removal of S disconnects G. Thus a cutset is also a cut. Note that the minimality property of a cutset implies that no proper subset of a cutset is a cutset.

Consider a spanning tree T of a connected graph G. Let b denote a branch of T. Removal of branch b disconnects T into two trees, T_1 and T_2. Let V_1 and V_2 denote the vertex sets of T_1 and T_2, respectively. Note that V_1 and V_2 together contain all the vertices of G. It is possible to verify that the cut $< V_1, V_2 >$ is a cutset of G and is called the **fundamental cutset** of G with respect to branch b of T. Thus, for a given graph G and a spanning tree T of G, we can construct $n-1$ fundamental cutsets, one for each branch of T. As an example, for the graph shown in Figure 2.5, the fundamental cutsets with respect to the spanning tree $T = [e_1, e_2, e_6, e_8]$ are the following:

Branch e_1: (e_1, e_3, e_4).
Branch e_2: (e_2, e_3, e_4, e_5).
Branch e_6: (e_6, e_4, e_5, e_7).
Branch e_8: (e_8, e_7).

Note that the fundamental cutset with respect to branch b contains b. Furthermore branch b is not present in any other fundamental cutset with respect to T.

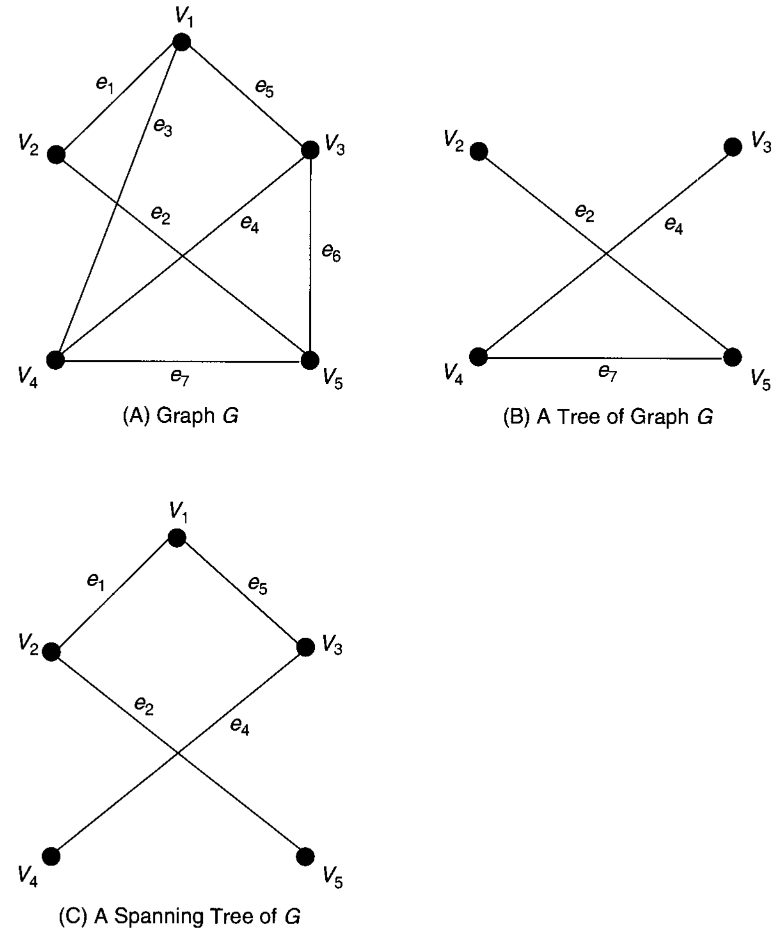


FIGURE 2.4 Examples of Tree Graphs

This section next identifies a special class of circuits of a connected graph G. Again let T be a spanning tree of G. Because exactly one path exists between any two vertices of T, adding a chord c to T produces a unique circuit. This circuit is called the **fundamental circuit** of G with respect to chord c of T. Note again that the fundamental circuit with respect to chord c contains c, and that the chord c is not present in any other fundamental circuit with respect to T. As an example, the set of fundamental circuits with respect to the spanning tree $T = (e_1, e_2, e_6, e_8)$ of the graph shown in Figure 2.5. is the following:

Chord e_3: (e_3, e_1, e_2).
Chord e_4: (e_4, e_1, e_2, e_6).
Chord e_5: (e_5, e_2, e_6).
Chord e_7: (e_7, e_8, e_6).

2.2.4 Incidence, Circuit, and Cut Matrices of a Graph

The incidence, circuit and cut matrices are coefficient matrices of Kirchhoff's voltage and current laws that describe an electrical network. This section defines these matrices and presents some properties of these matrices that are useful in studying electrical networks.

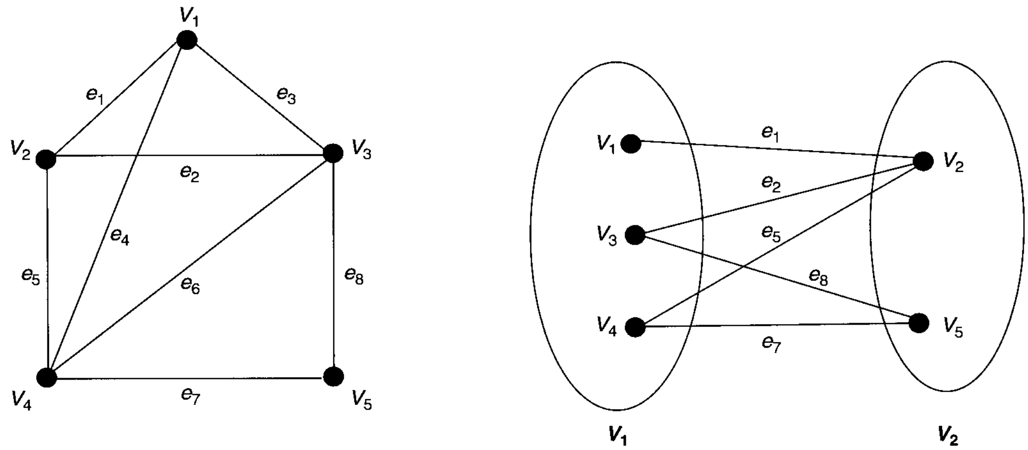
Incidence Matrix

Consider a connected directed graph G with n vertices and m edges and with no self-loops. The **all-vertex incidence matrix** $A_c = [a_{ij}]$ of G has n rows, one for each vertex, and m columns, one for each edge. The element a_{ij} of A_c is defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } j\text{th edge is incident out of the } i\text{th vertex.} \\ -1, & \text{if } j\text{th edge is incident into the } i\text{th vertex.} \\ 0, & \text{if the } j\text{th edge is not incident on the } i\text{th vertex.} \end{cases}$$

As an example, a graph and its A_c matrix are shown in Figure 2.6.

From the definition of A_c it should be clear that each column of this matrix has exactly two nonzero entries, one $+1$ and one -1, and therefore, any row of A_c can be obtained from the remaining rows. Thus,



(A) Graph G (B) Cut $<V_1, V_2>$ of G

FIGURE 2.5 A Graph and a Cut

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	1	0	0	0	0	1	−1
v_2	−1	1	0	0	0	0	0
v_3	0	−1	1	0	1	0	0
v_4	0	0	−1	−1	0	−1	0
v_5	0	0	0	1	−1	0	1

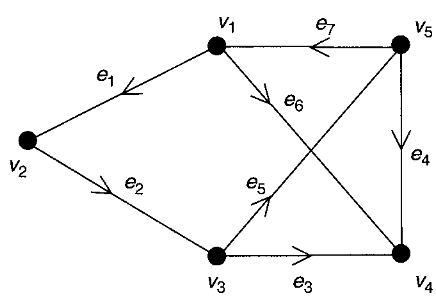


FIGURE 2.6 A Directed Graph

$$\text{rank}(A_c) \le n - 1.$$

An $(n-1)$ rowed submatrix of A_c is referred to as an **incidence matrix** of G. The vertex that corresponds to the row that is not in A_c is called the reference vertex of A.

Cut Matrix

Consider a cut (V_a, V_b) in a connected directed graph G with n vertices and m edges. Recall that $< V_a, V_b >$ consists of all those edges connecting vertices in V_a to V_b. This cut may be assigned an orientation from V_a to V_b or from V_b to V_a. Suppose the orientation of (V_a, V_b) is from V_a to V_b. Then the orientation of an edge (v_i, v_j) is said to agree with the cut orientation if $v_i \in V_a$, and $v_j \in V_b$.

The **cut matrix** $Q_c = [q_{ij}]$ of G has m columns, one for each edge and has one row for each cut. The element q_{ij} is defined as follows:

$$q_{ij} = \begin{cases} 1 & \text{if the } j\text{th edge is in the } i\text{th cut and its} \\ & \text{orientation agrees with the cut orientation.} \\ -1 & \text{if the } j\text{th edge is in the } i\text{th cut and its} \\ & \text{orientation does not agree with cut orientation.} \\ 0 & \text{if the } j\text{th edge is not in the } i\text{th cut.} \end{cases}$$

Each row of Q_c is called the **cut vector**. The edges incident on a vertex form a cut. Hence, the matrix A_c is a submatrix of Q_c. Next, another important submatrix of Q_c is identified.

Recall that each branch of a spanning tree T of connected graph G defines a fundamental cutset. The submatrix of Q_c corresponding to the $n-1$ fundamental cutsets defined by T is called the **fundamental cutset matrix** Q_f of G with respect to T.

Let $b_1, b_2, \ldots, b_{n-1}$ denote the branches of T. Assume that the orientation of a fundamental cutset is chosen to agree with that of the defining branch. Arrange the rows and the columns of Q_f so that the ith column corresponds to the fundamental cutset defined by b_i. Then the matrix Q_f can be displayed in a convenient form as follows:

$$Q_f = [U|Q_{fc}],$$

where U is the unit matrix of order $n-1$, and its columns correspond to the branches of T. As an example, the fundamental cutset matrix of the graph in Figure 2.6 with respect to the spanning tree $T = (e_1, e_2, e_5, e_6)$ is:

$$Q_f = \begin{bmatrix} e_1 & e_2 & e_5 & e_6 & e_3 & e_4 & e_7 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

It is clear that the rank of Q_f is $n-1$. Hence,

$$\text{rank}(Q_c) \geq n-1.$$

Circuit Matrix

Consider a circuit C in a connected directed graph G with n vertices and m edges. This circuit can be traversed in one of two directions, clockwise or counterclockwise. The direction chosen for traversing C is called the orientation of C. If an edge $e = (v_i, v_j)$ directed from v_i to v_j is in C, and if v_i appears before v_j in traversing C in the direction specified by the orientation of C, then the orientation agrees with the orientation of e.

The **circuit matrix** $B_c = [b_{ij}]$ of G has m columns, one for each edge, and has one row for each circuit in G. The element b_{ij} is defined as follows:

$$b_{ij} = \begin{cases} 1 & \text{if the } j\text{th edge is in the } i\text{th circuit and its} \\ & \text{orientation agrees with the circuit orientation.} \\ -1 & \text{if the } j\text{th edge is in the } i\text{th circuit and its} \\ & \text{orientation does not agree with circuit orientation.} \\ 0 & \text{if the } j\text{th edge is not in the } i\text{th circuit.} \end{cases}$$

The submatrix of B_c corresponding to the fundamental circuits defined by the chords of a spanning tree T is called **fundamental circuit matrix** B_f of G with respect to the spanning tree T.

Let $c_1, c_2, c_3, \ldots, c_{m-n+1}$ denote the chords of T. Arrange the columns and the rows of B_f so that the ith row corresponds to the fundamental circuit defined by the chord c_i and the ith column corresponds to the chord c_i. Then orient the fundamental circuit to agree with the orientation of the defining chord. The result is writing B_f as:

$$B_f = [U|B_{ft}],$$

where U is the unit matrix of order $m-n+1$, and its columns correspond to the chords of T.

As an example, the fundamental circuit matrix of the graph shown in Figure 2.6 with respect to the tree $T = (e_1, e_2, e_5, e_6)$ is given here.

$$B_f = \begin{matrix} & e_3 & e_4 & e_7 & e_1 & e_2 & e_5 & e_6 \\ e_3 & 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ e_4 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ e_7 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{matrix}$$

It is clear from this example that the rank of B_f is $m-n+1$. Hence,

$$\text{rank}(B_c) \geq m-n+1.$$

The following results constitute the foundation of the graph-theoretic application to electrical circuit analysis.

Theorem 2.4 (orthogonality relationship): (1) A circuit and a cutset in a connected graph have an even number of common edges; and (2) if a circuit and a cutset in a directed graph have $2k$ common edges, then k of these edges have the same relative orientation in the circuit and the cutset, and the remaining k edges have one orientation in the circuit and the opposite orientation in the cutset.

Theorem 2.5: If the columns of the circuit matrix B_c and the columns of the cut matrix Q_c are arranged in the same edge order, then

$$B_c Q_c^t = 0.$$

Theorem 2.6:

$$\text{rank}(B_c) = m-n+1.$$
$$\text{rank}(Q_c) = n-1.$$

Note that it follows from the above theorem that the rank of the circuit matrix is equal to the nullity of the graph, and the rank of the cut matrix is equal to the rank of the graph. This result, in fact, motivated the definitions of the rank and nullity of a graph.

2.3 Graphs and Electrical Networks

An electrical network is an interconnection of electrical network elements, such as resistances, capacitances, inductances, voltage, and current sources. Each network element is associ-

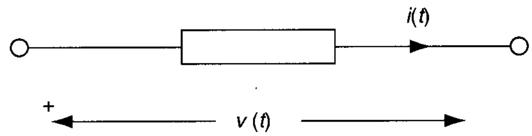


FIGURE 2.7 A Network Element with Reference Convention

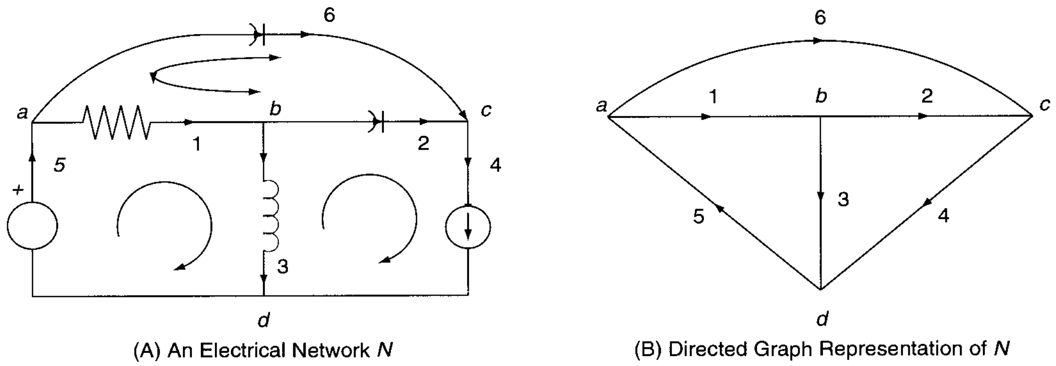


FIGURE 2.8 Graph Examples of KVL and KCL Equations

ated with two variables: the voltage variable $v(t)$ and the current variable $i(t)$. Reference directions are also assigned to the network elements as shown in Figure 2.7 so that $i(t)$ is positive whenever the current is in the direction of the arrow and so that $v(t)$ is positive whenever the voltage drop in the network element is in the direction of the arrow. Replacing each element and its associated reference direction by a directed edge results in the directed graph representing the network. For example, a simple electrical network and the corresponding directed graph are shown in Figure 2.8.

The physical relationship between the current and voltage variables of network elements is specified by Ohm's law. For voltage and current sources, the voltage and current variables are required to have specified values. The linear dependence among the voltage variables in the network and the linear dependence among the current variables are governed by Kirchhoff's voltage and current laws:

Kirchhoff's voltage law (KVL): The algebraic sum of the voltages around any circuit is equal to zero.

Kirchhoff's current law (KCL): The algebraic sum of the currents flowing out of a node is equal to zero.

As examples, the KVL equation for the circuit 1, 3, 5 and the KCL equation for vertex b in the graph of Figure 2.8 are the following:

Circuit 1, 3, 5. $V_1 + V_3 + V_5 = 0$
Vertex b: $-I_1 + I_2 + I_3 = 0$

It can be easily seen that KVL and KCL equations for an electrical network N can be conveniently written as:

$$A_c I_e = 0.$$
$$B_c V_e = 0.$$

A_c and B_c are, respectively, the incidence and circuit matrices of the directed graph representing N, I_e, and V_e, respectively, and these are the column vectors of element currents and voltages in N. Because each row in the cut matrix Q_c can be expressed as a linear combination of the rows of the matrix, in the above A_c can be replaced by Q_c. Thus, the result is as follows:

$$\text{KCL: } Q_c I_e = 0.$$
$$\text{KVL: } B_c V_e = 0.$$

Hence, KCL can also be stated as: the algebraic sum of the currents in any cut of N is equal to zero.

If a network N has n vertices and m elements and its graph is connected, then there are only $(n - 1)$ linearly independent cuts and only $(m - n + 1)$ linearly independent circuits. Thus, in writing KVL and KCL equations, only B_f and Q_f, respectively, need to be used. Thus, we have

$$\text{KCL: } Q_f I_e = 0$$
$$\text{KVL: } B_f V_e = 0$$

Note that KCL and KVL equations depend only on the way network elements are interconnected and not on the nature of the network elements. Thus, several results in electrical network theory are essentially graph-theoretic in nature. Some of those results of interest in electrical network analysis are presented in the remainder of this chapter. Note that for these results, a network N and its directed graph representation are both denoted by N.

2.3.1 Loop and Cutset Transformations

Let T be a spanning tree of an electrical network. Let I_c and V_t be the column vectors of chord currents and branch currents with respect to T.

1. **Loop transformation:**

$$I_e = B_f^t I_c.$$

2. **Cutset transformation:**

$$V_e = Q_f^t V_t.$$

If, in the cutset transformation, Q_f is replaced by the incidence matrix A, then we get the **node transformation** given below:

$$V_e = A^t V_n,$$

where the elements in the vector V_n can be interpreted as the voltages of the nodes with respect to the reference node r. (Note: The matrix A does not contain the row corresponding to the node r).

The above transformations have been extensively employed in developing different methods of network analysis. Two of these methods are described in the following section.

2.4 Loop and Cutset Systems of Equations

As observed earlier, the problem of network analysis is to determine the voltages and currents associated with the elements of an electrical network. These voltages and currents can be determined from Kirchhoff's equations and the element voltage–current (v–i) relations given by Ohm's law. However, these equations involve a large number of variables. As can be seen from the loop and cutset transformations, not all these variables are independent. Furthermore, in place of KCL equations, the loop transformation can be used, which involves only chord currents as variables. Similarly, KVL equations can be replaced by the cutset transformation that involves only branch voltage variables. Taking advantage of these transformations enables the establishment of different sytems of network equations known as the loop and cutset systems.

In deriving the loop system, the loop transformation is used in place of KCL in this case, the loop variables (chord currents) serve as independent variables. In deriving the cutset system, the cutset transformation is used in place of KVL, and the cutset variables (tree branch voltages) serve as the independent variables in this case. It is assumed that the electrical network N is connected and that N consists of only resistances, (R), capacitances (C), inductances (L), including mutual inductances, and independent voltage and current sources. It is also assumed that all initial inductor currents and initial capacitor voltages have been replaced by appropriate sources. Further, the voltage and current variables are all Laplace transforms of the complex frequency variable s. In N, there can be no circuit consisting of only independent voltage sources, for if such a circuit of sources were present, then by KVL, there would be a linear relationship among the corresponding voltages; this would violate the independence of the voltage sources. For the same reason, in N, there can be no cutset consisting of only independent current sources. Hence, there exists in N a spanning tree containing all the voltage sources but not current sources. Such a tree is the starting point for the development of both the loop and cutset systems of equations.

Let T be a spanning tree of the given network such that T contains all the voltage sources but no current sources. Partition the element voltage vector V_e and the element current vector I_e as follows:

$$V_e = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad \text{and} \quad I_e = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The subscripts 1, 2, and 3 refer to the vectors corresponding to the current sources, RLC elements, and voltage sources, respectively. Let B_f be the fundamental circuit matrix of N, and let Q_f be the fundamental cutset matrix of N with respect to T. The KVL and the KCL equations can then be written as follows:

$$\text{KVL: } B_f V_e = \begin{bmatrix} U & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0.$$

$$\text{KCL: } Q_f I_e = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & U \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = 0.$$

2.4.1 Loop Method of Network Analysis

Step 1: Solve the following for the vector I_l. Note that I_l is the vector of currents in the nonsource chords of T.

$$Z_l\, I_l = -B_{23} V_3 - B_{22} Z_2 B_{12} I_1, \tag{2.1}$$

where Z_2 is the impedance matrix of RLC elements and

$$Z_l = B_{22} Z_2 B_{22}.$$

Equation 2.1 is called the **loop system of equations.**

Step 2: Calculate I_2 using:

$$I_2 = B_{12}I_1 + B_{22}I_l. \tag{2.2}$$

Then

$$V_2 = Z_2 I_2. \tag{2.3}$$

Step 3: Determine V_1 and I_3 using the following:

$$V_1 = -B_{12}V_2 - B_{13}V_3. \tag{2.4}$$

$$I_3 = B_{13}I_1 + B_{23}I_l. \tag{2.5}$$

Note that I_1 and V_3 have specified values because they correspond to current and voltage sources, respectively.

2.4.2 Cutset Method of Network Analysis

Step 1: Solve the following for the vector V_b. Note that V_b is the vector of voltages in the nonsource branches of T.

$$Y_b\ V_b = -Q_{11}I_1 - Q_{12}Y_2Q_{22}V_b, \tag{2.6}$$

where Y_2 is the admittance matrix of RLC elements and

$$Y_b = Q_{12}Y_2Q_{12}$$

Equation 2.6 is called the **cutset system of equations.**
Step 2: Calculate V_2 using:

$$V_2 = Q_{12}V_b + Q_{22}V_3. \tag{2.7}$$

Then

$$I_2 = Y_2Y_2. \tag{2.8}$$

Step 3: Determine V_1 and I_3 using the following:

$$V_1 = Q_{21}V_b + Q_{21}V_3 \tag{2.9}$$

$$I_3 = -Q_{21}I_1 - Q_{22}I_2. \tag{2.10}$$

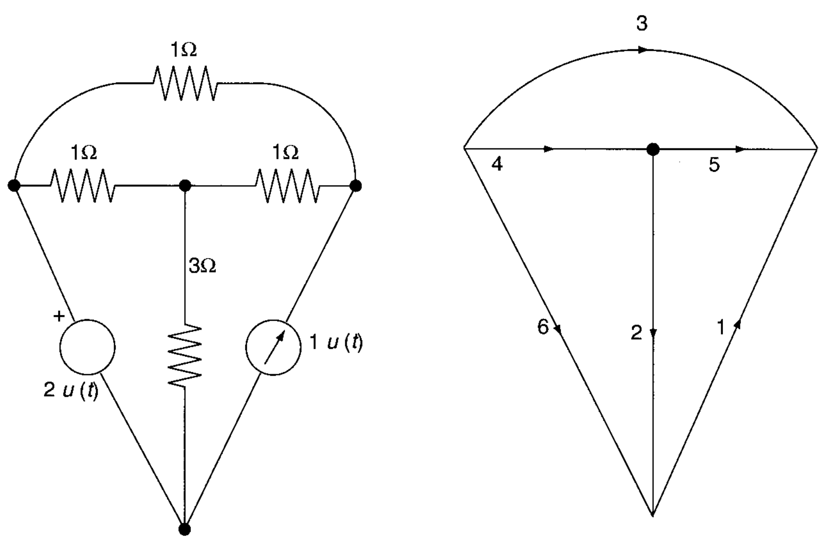
Note that I_1 and V_3 have specified values because they correspond to current and voltage sources.

This completes the cutset method of network analysis.

The following discussion illustrates the loop and cutset methods of analysis on the network shown in Figure 2.9(A). The graph of the network is shown in Figure 2.9(B). The chosen spanning tree T consists of edges 4, 5 and 6. Note that T contains the voltage source and has no current source. The fundamental circuit and the fundamental cutset matrices with respect to T are given below in the required partioned form:

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$$
$$B_f = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$$
$$Q_f = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



(A) Network (B) Graph

FIGURE 2.9 A Network and Its Graph

From these matrices results the following:

$$B_{12} = [0 \quad 0 \quad -1 \quad -1]$$
$$B_{13} = [1]$$
$$B_{22} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$
$$B_{23} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$Q_{11} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$Q_{12} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$
$$Q_{21} = [-1]$$
$$Q_{22} = [1 \quad 0 \quad 0 \quad 0].$$

The following also results:

$$Z_2 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$Y_2 = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$v_6 = 2\,\text{V}$$
$$i_1 = 1\text{A}$$

Loop Method

Edges 2 and 3 are nonsource chords. So,

$$I_l = \begin{bmatrix} i_2 \\ i_3 \end{bmatrix}$$

and substituting

$$Z_l = B_{22} Z_2 B_{22}^t = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

in equation 2.1 yields the following loop system of equations:

$$\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Solving for i_2 and i_3 yields:

$$I_l = \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = 1/11 \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

Using equation 2.2 results in:

$$I_2 = \begin{bmatrix} i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = 1/11 \begin{bmatrix} 7 \\ -5 \\ 1 \\ -6 \end{bmatrix}$$

Then using $V_2 = Z_2 I_2$ yields:

$$V_2 = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 1/11 \begin{bmatrix} 21 \\ -5 \\ 1 \\ -6 \end{bmatrix}$$

Finally, equations 2.4 and 2.5 yield the following:

$$V_1 = [v_1] = -27/11.$$
$$I_3 = [i_6] = 4/11$$

Cutset Method

Edges 4 and 5 are the nonsource branches. So,

$$V_b = \begin{bmatrix} v_4 \\ v_5 \end{bmatrix}$$

and substituting

$$Y_b = \begin{bmatrix} 7/3 & 1 \\ 1 & 2 \end{bmatrix}$$

in equation 2.6 yields the following cutset system of equations:

$$Y_b V_b = \begin{bmatrix} 7/3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1 \end{bmatrix}$$

Solving for V_b yields:

$$V_b = \begin{bmatrix} v_4 \\ v_5 \end{bmatrix} = 1/11 \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

From equation 2.7, the result is:

$$V_2 = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 1/11 \begin{bmatrix} 21 \\ -5 \\ 1 \\ -6 \end{bmatrix}$$

Then using $I_2 = Y_2 V_2$ yields:

$$I_2 = \begin{bmatrix} i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = 1/11 \begin{bmatrix} 7 \\ -5 \\ 1 \\ -6 \end{bmatrix}$$

Finally, using equations 2.9 and 2.10, the end results are:

$$V_1 = [v_1] = -27/11$$
$$I_3 = [i_6] = 4/11$$

This completes the illustration of the loop and cutset methods of circuit analysis.

Node Equations

Suppose a network N has no independent voltage sources. A convenient description of N with the node voltages as independent variables can be obtained as follows.

Let A be the incidence matrix of N with vertex v_r as reference. Let A be partitioned as $A = [A_{11}, A_{12}]$, where the columns of A_{11} and A_{12} correspond, respectively, to the RLC elements and current sources. If I_1 and I_2 denote the column vectors of RLC element currents and current source currents, then KCL equations for N become:

$$A_{11} I_1 = -A_{12} I_2.$$

By Ohm's law:

$$I_1 = Y_1 V_1,$$

where V_1 is the column vector of voltages of RLC elements and Y_1 is the corresponding admittance matrix. Furthermore, by the node transformation yields:

$$V_1 = A_{11}^t V_n,$$

where V_n is the column vector of node voltages. So the result from the KCL equations is the following:

$$(A_{11} Y_1 A_{11}^t) V_n = -A_{12} I_2.$$

The above equations are called **node equations**. The matrix $A_{11} Y_1 A_{11}^t$ is called the **node admittance matrix** of N.

2.5 Summary

This chapter has presented an introduction to certain basic results in graph theory and their application in the analysis of electrical circuits. For a more comprehensive treatment of other developments in circuit theory that are primarily of a graph-theoretic nature, refer to works by Swamy and Thulasiraman (1981) and Chen (1972). A publication by Seshu and Reed (1961) is an early work that first discussed many of the fundamental results presented in this chapter. A review by Watandabe and Shinoda (1999) is the most recent reference summarizing the graph theoretic contributions from Japanese circuit theory researchers.

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